The Ciência & Engenharia - Science & Engineering Journal
ISSN: 0103-944X
Volume 11 Issue 1, 2023
pp: 364 – 371

The Non-Homogenous Biquadratic Equation with Four Unknowns

$$xy(x+y)+30zw^3=0$$

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Abstract:

The Non-homogenous biquadratic equation with four unknowns given by $xy(x+y)+30zw^3=0$ is analyzed for obtaining different sets of non-zero distinct integer solutions through employing the linear transformations.

Keywords:

Non-homogenous biquadratic, Biquadratic with four unknowns , Integer solutions.

Introduction:

The theory of Diophantine equations offers a rich variety of fascinating problems. In particular, biquadratic diophantine equations, homogeneous and non-homogeneous have aroused the interest of numerous mathematicians since antiquity [1-5]. In this context, one may refer [6-17] for various problems on the biquadratic diophantine equations with four variables. However, often we come across non-homogeneous biquadratic equations and as such one may require its integral solution in its most general form. It is towards this end, this paper concerns with the problem of determining non-trivial integral solutions of the non-homogeneous equation with four unknowns given by $xy(x+y)+30zw^3=0$ is analyzed for obtaining different sets of non-zero distinct integer solutions through employing the linear transformations.

Notation :

$$t_{3,n}=\frac{n(n+1)}{2}$$

Method Of Analysis

The Non-homogenous biquadratic equation with four unknowns under consideration is,

$$xy(x+y)+30zw^3=0 \qquad (1)$$

Introduction oflinear Transformation

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$$\left.\begin{aligned} x &= u+v \\ y &= u-v \\ z &= u \end{aligned}\right\} \qquad (2)$$

in (1) leads to,

$$v^2 - u^2 = 15w^3 \qquad (3)$$

The above equation is solved for $u, v \,\&\, w$ through various methods. Substituting the values of u, v in (2) the corresponding integer solutions to the (1) are obtained.

The above process is illustrated below:

Method 1:

The above equation (3) is expressed as the system of double equations as given in Table: 1 below:

Table: 1

System	I	II	III	IV	V	VI	VII	VIII	IX	X	XI	XII
$v+u=$	$15w^3$	$5w^3$	$3w^3$	w^3	$15w^2$	$5w^2$	$3w^2$	w^2	$15w$	$5w$	$3w$	w
$v-u=$	1	3	5	15	w	$3w$	$5w$	$15w$	w^2	$3w^2$	$5w^2$	$15w^2$

Solution for System : I

Consider ,

$$v+u = 15w^3$$

$$v-u = 1$$

$$\Rightarrow v = \frac{15w^3+1}{2}, \quad u = \frac{15w^3-1}{2}$$

Let us take

$$w = 2k+1$$

$$\Rightarrow v = 60k^3 + 90k^2 + 45k + 8, \; u = 60k^3 + 90k^2 + 45k + 7$$

$$\therefore \; x = 120k^3 + 180k^2 + 90k + 15, y = 1, z = 60k^3 + 90k^2 + 45k + 7, w = 2k+1$$

Solution for System : II

Consider ,

$$v+u = 5w^3$$

$$v-u = 3$$

$$\Rightarrow v = \frac{5w^3+3}{2}, \quad u = \frac{5w^3-3}{2}$$

Let us take

$$w = 2k+1$$

$$\Rightarrow \mathrm{v} = 20\mathrm{k}^3 + 30\mathrm{k}^2 + 15\mathrm{k} + 4, \; u = 20k^3 + 30k^2 + 15k + 1$$

$$\therefore \; x = 40k^3 + 60k^2 + 30k + 5, y = -3, z = 20k^3 + 30k^2 + 15k + 1, w = 2k+1$$

Solution for System : III

Consider ,

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$v+u=3w^3$

$v-u=5$

$\Rightarrow v=\dfrac{3w^3+5}{2},\ \ u=\dfrac{3w^3-5}{2}$

Let us take

$w=2k+1$

$\Rightarrow \mathrm{v}=12\mathrm{k}^3+18\mathrm{k}^2+9\mathrm{k}+4,\ u=12k^3+18k^2+9k-1$

$\therefore\ x=24k^3+36k^2+18k+3, y=-5, z=12k^3+18k^2+9k-1, w=2k+1$

Solution for System: IV

Consider ,

$v+u=w^3$

$v-u=15$

$\Rightarrow v=\dfrac{w^3+15}{2},\ \ u=\dfrac{w^3-15}{2}$

Let us take

$w=2k+1$

$\Rightarrow \mathrm{v}=4\mathrm{k}^3+6\mathrm{k}^2+3\mathrm{k}+8,\ u=4k^3+6k^2+3k-7$

$\therefore\ x=8k^3+12k^2+6k+1, y=-15, z=4k^3+6k^2+3k-7, w=2k+1$

Solution for System: V

Consider ,

$v+u=15w^2$

$v-u=w$

$\Rightarrow v=\dfrac{15w^2+w}{2},\ \ u=\dfrac{15w^2-w}{2}$

$v=7w^2+t_{3,w},\ u=7w^2+t_{3,w-1}$

Let us take

$w=k$

$\Rightarrow \mathrm{v}=7\mathrm{k}^2+\mathrm{t}_{3,\mathrm{k}},\ u=7k^2+t_{3,k-1}$

$\therefore\ \mathrm{x}=15\mathrm{k}^2, y=-k, z=7k^2+t_{3,k-1}, w=k$

Solution for System: VI

Consider ,

$v+u=5w^2$

$v-u=3w$

$\Rightarrow v=\dfrac{5w^2+3w}{2},\ \ u=\dfrac{5w^2-3w}{2}$

$v=w^2+3t_{3,w},\ u=w^2+3t_{3,w-1}$

Let us take

$w=k$

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$\Rightarrow \mathrm{v} = \mathrm{k}^2 + 3\mathrm{t}_{3,\mathrm{k}}$, $u = k^2 + 3t_{3,k-1}$

$\therefore\ \mathrm{x} = 5\mathrm{k}^2, y = -3k, z = k^2 + 3t_{3,k-1}, w = k$

Solution for System: VII

Consider ,

$v + u = 3w^2$

$v - u = 5w$

$$\Rightarrow v = \frac{3w^2 + 5w}{2}, \quad u = \frac{3w^2 - 5w}{2}$$

$v = w + 3t_{3,w}$, $u = -w + 3t_{3,w-1}$

Let us take

$w = k$

$\Rightarrow \mathrm{v} = \mathrm{k} + 3\mathrm{t}_{3,\mathrm{k}}$, $u = -k + 3t_{3,k-1}$

$\therefore\ \mathrm{x} = 3\mathrm{k}^2, y = -5k, z = -k + 3t_{3,k-1}, w = k$

Solution for System: VIII

Consider ,

$v + u = w^2$

$v - u = 15w$

$$\Rightarrow v = \frac{w(w+15)}{2}, \quad u = \frac{w(w-15)}{2}$$

$v = 7w + t_{3,w}$, $u = -7w + 3t_{3,w-1}$

Let us take

$w = k$

$\Rightarrow \mathrm{v} = 7\mathrm{k} + \mathrm{t}_{3,\mathrm{k}}$, $u = -7k + t_{3,k-1}$

$\therefore\ \mathrm{x} = \mathrm{k}^2, y = -15k, z = -7k + t_{3,k-1}, w = k$

Solution for System: IX

Consider ,

$v + u = 15w$

$v - u = w^2$

$$\Rightarrow v = \frac{w(15+w)}{2}, \quad u = \frac{w(15-w)}{2}$$

$v = 7w + t_{3,w}$, $u = 7w - t_{3,w-1}$

Let us take

$w = k$

$\Rightarrow \mathrm{v} = 7\mathrm{k} + \mathrm{t}_{3,\mathrm{k}}$, $u = 7k - t_{3,k-1}$

$\therefore\ \mathrm{x} = 15\mathrm{k}, y = -k^2, z = 7k - t_{3,k-1}, w = k$

Solution for System: X

Consider ,

$v + u = 5w$

$v - u = 3w^2$

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$$\Rightarrow v = \frac{5w+3w^2}{2}, \quad u = \frac{5w-3w^2}{2}$$

$v = w + 3t_{3,w}, \; u = w - 3t_{3,w-1}$

Let us take

$$w = k$$

$v = k + 3t_{3,k}, \; u = k - 3t_{3,k-1}$

$\therefore \; \mathrm{x} = 5\mathrm{k}, y = -3k^2, z = k - 3t_{3,k-1}, w = k$

Solution for System: XI

Consider ,

$v + u = 3w$

$v - u = 5w^2$

$$\Rightarrow v = \frac{3w+5w^2}{2}, \quad u = \frac{3w-5w^2}{2}$$

$v = w^2 + 3t_{3,w}, \; u = -w^2 - 3t_{3,w-1}$

Let us take

$$w = k$$

$\Rightarrow \mathrm{v} = \mathrm{k}^2 + 3\mathrm{t}_{3,\mathrm{k}}, \; u = -k^2 - 3t_{3,k-1}$

$\therefore \; \mathrm{x} = 3\mathrm{k}, y = -5k^2, z = -k^2 - 3t_{3,k-1}, w = k$

Solution for System: XII

Consider ,

$v + u = w$

$v - u = 15w^2$

$$\Rightarrow v = \frac{w+15w^2}{2}, \quad u = \frac{w-15w^2}{2}$$

$v = 7w^2 + t_{3,w}, \; u = -7w^2 - t_{3,w-1}$

Let us take

$$w = k$$

$\Rightarrow \mathrm{v} = 7\mathrm{k}^3 + \mathrm{t}_{3,\mathrm{k}}, \; \mathrm{u} = 7\mathrm{k}^2 - \mathrm{t}_{3,\mathrm{k}-1}$

$\therefore \; \mathrm{x} = \mathrm{k}, y = -15k^2, z = -7k^2 - t_{3,k-1}, w = k$

Method: 2

Taking

$$v = ku, (k > 1) \tag{4}$$

in (3), we have

$\left(k^2 - 1\right)u^2 = 15w^3$

which is satisfied by,

$$\left.\begin{aligned} u &= 15^2\left(k^2-1\right)\alpha^{3s} \\ w &= 15\left(k^2-1\right)\alpha^{2s} \end{aligned}\right\} \tag{5}$$

and from (4),

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$$v = 15^2\left(k^2 - 1\right)k.\alpha^{3s}$$

Substitute the above values of $u \,\&\, v$ in (2), one has

$$x = (k+1).15^2\left(k^2 - 1\right).\alpha^{3s} \quad (6)$$

$$y = -(k-1).15^2\left(k^2 - 1\right).\alpha^{3s} \quad (7)$$

$$z = 15^2\left(k^2 - 1\right).\alpha^{3s} \quad (8)$$

where, $k > 1$

Thus (5),(6),(7)& (8) represents the integer solutions to (1).

Method : 3

Taking

$$w = -v \quad (9)$$

in (3) ,it is written as,

$$u^2 = v^2\left(15v + 1\right) \quad (10)$$

Assume

$$\alpha^2 = 15v + 1 \quad (11)$$

whose smallest positive integer solution is given by,

$$v = v_0 = 1,\ \alpha = \alpha_0 = 4 \quad (12)$$

Let v_1, α_1 be the 2^{nd} integer solution to (10) given by,

$$\left.\begin{aligned} v_1 &= 2\alpha_0 + 15 + v_0 \\ \alpha_1 &= \alpha_0 + 15 \end{aligned}\right\} \quad (13)$$

Using the above values in (10) & simplifying we get,

$$h = 2\alpha_0 + 15$$

where h is the unknown integer to be determined.

in view of (13), we have

$$v_1 = 2\alpha_0 + 15 + v_0$$

$$\alpha_1 = \alpha_0 + 15$$

Following the above procedure, one obtains

$$v_2 = 2\alpha_1 + 15 + v_1$$

$$\alpha_2 = \alpha_1 + 15$$

The repetition of the above process leads to the general solution v_k, α_k of (10) given by,

$$\left.\begin{aligned} v_k &= 2k\alpha_0 + 15k^2 + v_0 \\ \text{(i.e)} \quad v_k &= 15k^2 + 8k + 1 \\ \alpha_k &= 15k + 4 \end{aligned}\right\}$$

(14)

Substituting (14) in (9) ,(10) and (2), we have

$$x_k = (15k^2 + 8k + 1)(15k + 5)$$
$$y_k = (15k^2 + 8k + 1)(15k + 3)$$
$$z_k = (15k^2 + 8k + 1)(15k + 4)$$
$$w_k = -(15k^2 + 8k + 1)$$

where $k \geq 0$.

Conclusion:

In this paper, an attempt has been made to determine non-zero distinct integer solutions to the non-homogeneous biquadratic equation with four unknowns given by $xy(x+y)+30zw^3 = 0$. The researchers in this field may search for other sets of integer solutions to the equation under consideration and other forms of biquadratic equations with four or more variables.

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