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On Finding Integer Solutions to Sextic Equation with Three Unknowns

$$x^2 + y^2 = 8z^6$$

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Abstract

This paper deals with the problem of finding non-zero distinct integer solutions to the non-homogeneous ternary sextic equation given by $x^2 + y^2 = 8z^6$. A few interesting relations between the solutions and special numbers are exhibited.

Key words: non-homogeneous sextic ,ternary sextic ,integer solutions

Notations:

$t_{3,n}$ =Triangular number of rank n

P_n^3 =Triangular pyramidal number of rank n

P_n^5 =Pentagonal pyramidal number of rank n

P_n^6 =Hexagonal pyramidal number of rank n

CP_a^6 =Centered Hexagonal pyramidal number of rank a

CP_a^{12} =Centered Dodecagonal pyramidal number of rank a

CP_a^9 =Centered Nonagonal pyramidal number of rank a

$Ct_{a,16}$ =Centered hexadecogonal number of rank a

$Ct_{a,24}$ =Centered icositetragonal number of rank a

Introduction

It is well-known that a diophantine equation is an algebraic equation with integer coefficients Involving two or more unknowns such that the only solutions focused are integer solutions .No doubt that diophantine equations are rich in variety [1-4] .There is no universal method available to know whether a diophantine equation has a solution or finding all solutions if it exists.For equations with more than three variables and degree at least three, very little is known.It seems that much work has not been done in solving higher degree diophantine

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equations.While focusing the attention on solving sextic diophantine equations with variables at least three,the problems illustrated in [5-22] are observed. This paper focuses on finding integer solutions to the sextic equation with three unknowns $x^2+y^2=8z^6$.A few interesting relations between the solutions and special numbers are exhibited.

Method of analysis:

The non-homogeneous Diophantine equation of degree six with three unknowns to be solved in integers is

$$x^2+y^2=8z^6 \tag{1}$$

Different ways of determining non-zero distinct integer solutions to (1) are illustrated below:

Way:1

Introduction of the transformations

$$x=m(m^2+n^2),\, y=n(m^2+n^2) \tag{2}$$

in (1) leads to

$$m^2+n^2=2z^2 \tag{3}$$

Assume

$$z=a^2+b^2 \tag{4}$$

Write 2 on the R.H.S. of (3) as

$$2=(1+i)(1-i) \tag{5}$$

Using (4) & (5) in (3) and employing the method of factorization ,define

$$m+in=(1+i)(a+ib)^2$$

from which ,on equating the real and imaginary parts,one obtains

$$m=a^2-b^2-2ab, n=a^2-b^2+2ab \tag{6}$$

From (2) and (6), one has

$$\left.\begin{aligned} x&=2(a^2+b^2)^2(a^2-b^2-2ab),\\ y&=2(a^2+b^2)^2(a^2-b^2+2ab)\end{aligned}\right\} \tag{7}$$

Thus (4) and (7) represent the non-zero distinct integer solutions to (1).

Properties:

1. $x+y$ is written as the difference of two squares
2. $\dfrac{y-x}{z^2}=16t_{3,b}$ when $a=b+1$
3. $\dfrac{y-x}{z^2}=16\,P_b^5$ when $a=b(b+1)$
4. $\dfrac{y-x}{z^2}=48\,P_{b-1}^3$ when $a=(b-1)(b+1)$

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pp: 350 – 355

5. $\frac{y^2 - x^2}{2}$ is a nasty number when $a \geq b \geq 0$

6. $\frac{y - x}{z^2} = 24P_n^6$ when $a = t_{3,n}$ $, b = 4n - 1$

7. $\frac{y - x}{z^2} = 8CP_a^6$ when $b = a^2$

8. $\frac{y - x}{z^2} = 8CP_a^{12}$ when $b = 2a^2 - 1$

9. $\frac{y - x}{z^2} = 16CP_a^9$ when $b = 3a^2 - 1$

10. $\frac{y - x}{z^2} = Ct_{a,16} - 1$ when $b = a + 1$

11. $\frac{y - x}{z^2} = 2(Ct_{a,24} - 1)$ when $b = 3(a + 1)$

Note:1:

As (1) is symmetric with respect to x,y and z, The triples given by
(x,-y,z) ,(x,y,-z) ,(-x,y,z) ,(-x,y,-z) ,(-x,-y,-z) ,(-x,-y,z) ,(x,-y,-z)
also satisfy (1).

Note:2

One may also express 2 on the R.H.S. of (3) as below:

$$2 = \frac{(7+i)(7-i)}{25}, 2 = \frac{(1+7i)(1-7i)}{25}$$

The repetition of the above process exhibits two more distinct integer solutions to (1).

Note:3

It is noted that ,in addition to (2),the transformations given by

$$x = m(m^2 - 3n^2),\ y = n(3m^2 - n^2)$$

Also reduces (1) to (3).Following the above procedure ,different sets of integer solutions to (1) are obtained.

Way:2

(3) is written as

$$m^2 + n^2 = 2z^2 * 1 \tag{8}$$

Write 1 on the R.H.S. of (8) as

$$1 = \frac{(r^2 - s^2 + i2rs)(r^2 - s^2 - i2rs)}{(r^2 + s^2)^2} \tag{9}$$

Substituting (4),(5) and (9) in (8) and employing the method of factorization,define

$$m + in = \frac{(1+i)(r^2 - s^2 + i2rs)(a+ib)^2}{(r^2 + s^2)}$$

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On equating the real and imaginary parts and taking

$$a=(r^2+s^2)A, b=(r^2+s^2)B \tag{10}$$

One obtains

$$\left.\begin{aligned} m&=(r^2+s^2)\left[(r^2-s^2)(A^2-B^2-2AB)-2rs(A^2-B^2+2AB)\right],\\ n&=(r^2+s^2)\left[(r^2-s^2)(A^2-B^2+2AB)+2rs(A^2-B^2-2AB)\right] \end{aligned}\right\} \tag{11}$$

Substituting (10) and (11) in (4),(2),the corresponding non-zero distinct integer solutions to (1) are obtained.

Way:3

Introducing the transformations

$$x=p^2-q^2+2pq \,, y=p^2-q^2-2pq \tag{12}$$

in (1),it simplifies to

$$p^2+q^2=2z^3 \tag{13}$$

Substituting (4) & (5) in (13) and employing the method of factorization define

$$p+iq=(1+i)(a+ib)^3$$

On equating the real and imaginary parts, note that

$$p=a^3-3ab^2-3a^2b+b^3 \,, q=a^3-3ab^2+3a^2b-b^3 \tag{14}$$

Using (14) in (12),one has

$$\left.\begin{aligned} x&=2\left[(a^3-3ab^2-3a^2b+b^3)^2-2(3a^2b-b^3)^2\right]\\ y&=-2\left[(a^3-3ab^2+3a^2b-b^3)^2-2(3a^2b-b^3)^2\right] \end{aligned}\right\} \tag{15}$$

Thus,(4) and (15) represent the integer solutions to (1).

Note:4

Taking into consideration Note:2 and Way:2 ,two more sets of integer solutions to (1) are determined.

Conclusion:

In this paper ,an attempt has been made to determine the non-zero distinct integer solutions to the non-homogeneous ternary sextic diophantine equation given in the title through employing transformations. The researchers in this area may search for other choices of transformations to obtain integer solutions to the ternary sextic diophantine equation under consideration.

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