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Unique Isolated Signed Dominating Function in Graphs

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Abstract
A graph G is said to be an unique isolated signed dominating function (UISDF) if $f: V(G) \to \{-1, +1\}$, there exists exactly one vertex $w \in V(G)$ with $f(N[w]) = +1$. The unique isolated signed domination number $\gamma_{is}^{u}(G)$, is the minimum weight of an UISDF of the graph G. Some properties of unique isolated signed dominating function of some disconnected graphs and special graphs were presented in this paper.
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Key Words: isolated domination, signed dominating function, isolated signed dominating function, unique isolated signed dominating function.

1 Introduction

In this paper, only finite graphs, simple graphs and undirected graphsare considered for presenting. The set of vertices is denoted by $V(G)$, edges is denoted by $E(G)$ and thus graph G is denoted by $G = (V(G), E(G)))$. The notations are followed by Harary [3] for a general reference on graph theory.

Let the open neighborhood of any vertex, $v \in V(G)$ is $N_G(v) = \{u \in V(G): uv \in E(G)\}$ Let the closed neighborhood of any vertex, $v \in V(G)$ is $N_G[v] = \{v\} \cup N(v)$. Then, the degree of v is $deg_G(v) = |N_G(v)|$. The minimum degree of G is defined by $\delta(G) = \min_{v \in V(G)} \{deg(v)\}$ and the maximum degree of G is defined by $\Delta(G) = \max_{v \in V(G)} \{deg(v)\}$. Any vertex $v \in V(G)$ of degree one is called as a pendent vertex. A vertex $v \in V(G)$ adjacent to any pendent vertex is called as a stem.

A dominating function of any graph G is a function $f: V(G) \to \{0,1\}$ if for every vertex $v \in V(G)$, $f(N[v]) \geq 1$ [4].

In 1995, the concept of signed dominating function in graphs is defined by J.E.Dunbar et al. [2]. A signed dominating function of any G, is defined by a function $f: V(G) \to \{-1, +1\}$, if for every vertex $v \in V(G)$, $f(N[v]) \geq 1$. The minimum weight of a signed dominating function is the signed domination number, $\gamma_s(G)$ [2].

In 2012, Changping Wang [1], used the definition of signed dominating function and introduced a new domination parameter as signed k-dominating function. For any integer $k \geq$

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1, a signed k-dominating function is defined as a function $f: V(G) \rightarrow \{-1, +1\}$ satisfying $\sum_{w \in N[v]} f(w) \geq k$ for every $v \in V(G)$. The signed k-domination number is the minimum of the values of $\sum_{v \in V(G)} f(v)$ taken over all signed k-dominating functions f, denoted by $\gamma_{ks}(G)$.

In 2016, the concept of isolate domination is introduced by Hameed and Balamurugan [7]. An isolate dominating set is a dominating set S of a graph G, if $< S >$ has at least one isolated vertex [7]. An isolate dominating set S is said to be minimal if no proper subset of S is an isolate dominating set. The minimum cardinality of a minimal isolate dominating set of G is called as the isolate domination number $\gamma_0(G)$ and the maximum cardinality of a minimal isolate dominating set of G is called as the upper isolate domination number $\Gamma_0(G)$.

In 2019, S. Rishitha Dayana and S. Chandra Kumar [6], introduced a new domination parameter called isolated signed dominating function. An isolated signed dominating function (ISDF) of any graph G is defined as a signed dominating function such that $f(N[w]) = +1$ for at least one vertex w. The weight of f is the sum of the values $f(v)$ for all $v \in V(G)$ and is denoted by $w(f)$. An isolated signed domination number $\gamma_{is}(G)$, is the minimum weight of an ISDF of G.

In continuation to isolated signed dominating function (ISDF), we introduced a new parameter in signed domination known as unique isolated signed dominating function(UISDF) in graphs. An UISDF of a graph G is a signed dominating function such that there exists exactly one vertex $w \in V(G)$ with $f(N[w]) = +1$. The minimum weight of an UISDF of G is the unique isolated signed domination number of G, $\gamma_{is}^{u}(G)$. In this paper, some properties of UISDF and unique isolated signed domination number of some graphs were presented.

2 Main Results

Lemma 1. *For any graph G, we have $\gamma_s(G) \leq \gamma_{is}(G) \leq \gamma_{is}^{u}(G)$.*

Proof. Since every ISDF is a SDF and every UISDF is an ISDF, we have $\gamma_s(G) \leq \gamma_{is}(G) \leq \gamma_{is}^{u}(G)$.

Definition 2. *[1] For any integer $k \geq 1$, a signed k-dominating function is defined as a function $f: V(G) \rightarrow \{-1, +1\}$ satisfying $\sum_{w \in N[v]} f(w) \geq k$ for every vertex $v \in V(G)$. The signed k-domination number, $\gamma_{ks}(G)$ is the minimum of the values of $\sum_{v \in V(G)} f(v)$ taken over all signed k-dominating functions f.*

Theorem 3. *Let $n \geq 2$ be any integer and G be a disconnected graph with n components such that the first $r (\geq 1)$ components admit UISDF. Then $\gamma_{is}^{u}(G) = \min_{1 \leq i \leq r}\{t_i\}$, where $t_i = \gamma_{is}^{u}(G_i) + \sum_{j=1, j \neq i}^{n} \gamma_{2s}(G_j)$.*

Proof. Let $G_1, G_2, \ldots, G_n$ be the graphs *with n components and* $G_1, G_2, \ldots, G_r$ be the components which admit UISDF. With out loss of generality, let us assume that $t_1 = \min_{1 \leq i \leq r}\{t_i\}$.

Let us assume that f_1 as minimum UISDF of G_1 and f_i be the minimum S2DF of G_i for $2 \leq i \leq n$. Then $f: V(G) \rightarrow \{-1, +1\}$ is defined by $f(x) = f_i(x)$, $x \in V(G_i)$, is an UISDF of G with weight $\gamma_{is}^{u}(G_1) + \sum_{i=2}^{n} \gamma_{2s}(G_i)$ and so $\gamma_{is}^{u}(G) \leq \gamma_{is}^{u}(G_1) + \sum_{i=2}^{n} \gamma_{2s}(G_i) = t_1$.

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Let us assume that g as minimum UISDF of G. Then there exists an integer j such that $g|_{G_j}$ is a minimum UISDF of G_j for some j with $1 \leq j \leq r$. Also for each i with $1 \leq i \leq n (i \neq j)$, $g|_{G_i}$ is a minimum S2DF of G_i. Therefore $w(g) \geq \gamma_{is}^u(G_j) + \sum_{i=1, i\neq j}^{n} \gamma_{2s}(G_i) = t_j \geq t_1$ and hence $\gamma_{is}^u(G) = \min_{1\leq i\leq r}\{t_i\}$.

Theorem 4. *Let H be any graph of order n which does not admit UISDF. Then $G = H \cup rK_1 (r \geq 1)$ admit UISDF if and only if $r = 1$. In this case $\gamma_{is}^u(G) = 1 + \gamma_{2s}(H)$*

Proof. Suppose there exists an UISDF of G, say 'f'. Let $V(G) = V(H) \cup V(rK_1)$, where $V(H) = \{u_1, u_2, \ldots, u_n\}$ and $V(rK_1) = \{v_i : i = 1,2, \ldots, r\}$. Suppose $r \geq 2$. Then the vertices v_1 and v_2 must have $+1$ sign. In this case, $f(N[v_1]) = f(N[v_2]) = +1$, which is a contradiction to f is UISDF. Thus $r = 1$.

Coversely, suppose $r = 1$. Let us define a function $f: V(G) \to \{-1, +1\}$ by $f(u) = 1$ for all $u \in V(G)$. Then $f(N[v_1]) = 1$ and $f(N[u_i]) \geq 2$ for $2 \leq i \leq n$. The graph G admit UISDF.

By taking $r = 1$ in Theorem 3, we can have $\gamma_{is}^u(G) = 1 + \gamma_{2s}(H)$.

Lemma 5. *Any odd regular graph G does not admit UISDF.*

Proof. Since $|N[v]|$ is even for all $v \in V(G)$, $f(N[v]) \neq 1$ for any UISDF $f: V \to \{-1, +1\}$ and for any vertex $v \in V(G)$.

Definition 6. *[4] A set S is a k-dominating set if for every vertex $v \in V(G) - S$, $|N(u) \cap S| \geq k$. The k-domination number, $\gamma_k(G)$ is the minimum weight of a k-dominating set.*

Theorem 7. *Let G be a connected graph of order $n \geq 2$ which admits UISDF. Then $2\gamma_2(G) - n \leq \gamma_{is}^u(G)$.*

Proof. Let f be a minimum UISDF of G, $V^+ = \{u \in V: f(u) = +1\}$ and $V^- = \{v \in V: f(v) = -1\}$. If $V^- = \phi$, then G does not admit UISDF. Thus $V^- \neq \phi$.

If $w \in V^-$, then w has at least two neighbors in V^+. Therefore the graph G, V^+ is a 2-dominating set for G and so $|V^+| \geq \gamma_2(G)$. Since $\gamma_{is}^u(G) = |V^+| - |V^-|$ and $n = |V^+| + |V^-|$, we have $\gamma_{is}^u(G) = 2|V^+| - n$. Thus $\gamma_{is}^u(G) \geq 2\gamma_2(G) - n$.

Theorem 8. *For any graph G, maximum degree Δ and minimum degree δ, then $\gamma_{is}^u(G) \geq \frac{2+(\delta-\Delta)n}{\Delta+\delta+2}$.*

Proof. Let f be a minimum UISDF of G. Since $|V^+| + |V^-| = n$ and $|V^+| - |V^-| = \gamma_{is}^u(G)$, we get $|V^+| = \frac{n+\gamma_{is}^u(G)}{2}$ and $|V^-| = \frac{n-\gamma_{is}^u(G)}{2}$. By definition of UISDF of G, $f(N[v]) \geq 1$ for all $v \in V(G)$. Then $\sum_{v\in V(G)} (d(v) + 1)f(v) \geq 1$. Therefore, $\sum_{v\in V^+} (d(v))f(v) + \sum_{v\in V^-} (d(v))f(v) + \gamma_{is}^u(G) \geq 1$. That is, $\Delta|V^+| - \delta|V^-| + \gamma_{is}(G) \geq 1$ and so $\frac{(n+\gamma_{is}^u(G))\Delta}{2} - \frac{(n-\gamma_{is}^u(G))\delta}{2} + \gamma_{is}^u(G) \geq 1$. Hence $\gamma_{is}^u(G) \geq \frac{2+(\delta-\Delta)n}{\Delta+\delta+2}$.

Proposition 9. *For an integer $n \geq 3$, the cycle $G = C_n$ does not admit UISDF.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{v_i v_{i+1} / 1 \leq i \leq n-1\} \cup \{v_n v_1\}$. Suppose there exist an UISDF of G, say 'f'. Suppose $f(v_i) = -1$ for some $1 \leq i \leq n$, without loss of generality, let it be v_2. Since $N[v_2] = \{v_1, v_2, v_3\}$ and $f(N[v_2]) = 1$, we

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must have $f(v_1) = f(v_3) = 1$. Since $N[v_3] = \{v_2, v_3, v_4\}$ and $f(N[v_3]) \geq 1$, we must have $f(v_4) = 1$. In this case $f(N[v_i]) = 1$, for $i = 2$ and $i = 3$, a contradiction to the definition of UISDF.

Thus $f(v_i) = 1$ for all $1 \leq i \leq n$ and so $f(N[u]) = 3$ for all $u \in V(G)$, a contradiction to the definition of UISDF.

Proposition 10. *For any integer, $n \geq 2$, the graph $G = P_n$ does not admit UISDF.*

Proof. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $E(G) = \{v_i v_{i+1} / 1 \leq i \leq n-1\}$. Suppose there exists an UISDF of G, say 'f'. Suppose $f(v_i) = -1$ for some i with $2 \leq i \leq n-1$. Then $f(N[v_{i-1}]) = f(N[v_{i-1}]) = 1$, a contradiction.

Suppose $f(v_1) = -1$ or $f(v_n) = -1$, then $f(N[v_1]) < 1$ or $f(N[v_n]) < 1$ respectively, a contradiction.

Thus $f(v_i) = 1$ for all $1 \leq i \leq n$, so $f(N[u]) \geq 2$ for all $u \in V(G)$, a contradiction to the definition of UISDF.

Definition 11. *[5] $P_n^{(2)+}$ is a graph obtained from P_n by joining the internal vertices v to the one end v_1 such that $d(v, v_1)$ is even. Then number of vertices is n and the number of edges is $n - 1 + \left\lfloor \frac{n-1}{2} \right\rfloor$.*

Lemma 12. *Let $n \geq 5$ be an odd integer. Then the graph $G = P_n^{(2)+}$ admits UISDF with $\gamma_{is}^u(G) = n - 2$.*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. Let f be an UISDF. Then $f(N[v]) \geq 1$ for all $v \in V(G)$.

Suppose $f(v_1) = -1$, then $f(N[v_2]) = f(N[v_n]) = 1$, a contradiction.

Suppose $f(v_n) = -1$, then $f(N[v_n]) = f(N[v_{n-1}]) = 1$, a contradiction.

Suppose $f(v_i) = -1$ for some $i = 3,5, \ldots, n-2$, then $f(N[v_{i-1}]) = f(N[v_{i+1}]) = 1$, a contradiction.

Suppose $f(v_i) = f(v_j) = -1$ for some $i \neq j$, $i, j \in \{2,4, \ldots, n-1\}$. In this case, $f(N[v_i]) = f(N[v_j]) = 1$, a contradiction.

Therefore $f(v) = -1$ for a maximum of one vertex in G. Thus $w(f) = n - 2$ and so $\gamma_{is}^u(G) \geq n - 2$.

Let us define a function $f: V(G) \to \{-1, +1\}$ as follows:

$$f(v_i) = \begin{cases} -1 & when\ i = 2 \\ +1 & otherwise. \end{cases}$$

In this case, $f(N[v_2]) = 1$ and $f(N[v_i]) \geq 2$ for all $i \neq 2$. Thus f is an UISDF with $w(f) = n - 2$ and hence $\gamma_{is}^u(G) \leq n - 2$.

Lemma 13. *Let $n \geq 6$ be an even integer. Then the graph $G = P_n^{(2)+}$ admits UISDF with $\gamma_{is}^u(G) = n - 2$.*

Proof. Let $\{v_1, v_2, \ldots, v_n\}$ be the vertex set of G. Let f be an UISDF. Then $f(N[v]) \geq 1$ for all $v \in V(G)$. Suppose $f(v_{n-1}) = f(v_n) = -1$, then $f(N[v_n]) = 0$, a contradiction.

Suppose $f(v_i) = -1$ for some $i = 3,5, \ldots, n-3$, then $f(N[v_{i-1}]) = f(N[v_{i+1}]) = 1$, a contradiction.

Suppose $f(v_i) = f(v_j) = -1$ for some $i \neq j$, $i, j \in \{2,4,\ldots,n-2\}$, then $f(N[v_i]) = f(N[v_j]) = 1$, a contradiction.

Suppose $f(v_1) = -1$.

Case 1: If $f(v_2) = -1$, then $f(N[v_2]) \leq -1$, a contradiction.

Case 2: If $f(v_i) = -1$ for some $i \in \{4,6,\ldots,n-2\}$, then $f(N[v_2]) = f(N[v_i]) = 1$, a contradiction.

Therefore $f(v) = -1$ for a maximum of one vertex in G. Thus $w(f) = n-2$ and so $\gamma_{is}^u(G) \geq n-2$.

Let us define a function $f: V(G) \rightarrow \{-1,+1\}$ as follows:

$$f(v_i) = \begin{cases} -1 & when\ i = 1 \\ +1 & otherwise. \end{cases}$$

In this case, $f(N[v_2]) = 1$ and $f(N[v_i]) \geq 2$ for all $i \neq 2$. Thus f is an UISDF with $w(f) = n-2$ and hence $\gamma_{is}^u(G) \leq n-2$.

From Lemma 12 and Lemma 13, we can have the following theorem.

Theorem 14. *Let $n \geq 5$ be an integer. Then the graph $G = P_n^{(2)+}$ admits UISDF with $\gamma_{is}^u(G) = n-2$.*

Remark 15. *Let G be a graph which admits UISDF. If $u \in V(G)$ and u is a pendent vertex adjecent to another vertex ' w '. Then $f(u) = f(w) = +1$ (otherwise $f(N[u]) \leq 0$).*

Definition 16. *[5] The graph $\langle K_{1,m}: K_{1,n} \rangle$ is a graph obtained from a (m,n) bistar by subdividing the middle edge with a new vertex (as given in figure : 1). Note that $|V(\langle K_{1,m}: K_{1,n} \rangle)| = m+n+3$ and $|E(\langle K_{1,m}: K_{1,n} \rangle)| = m+n+2$.*

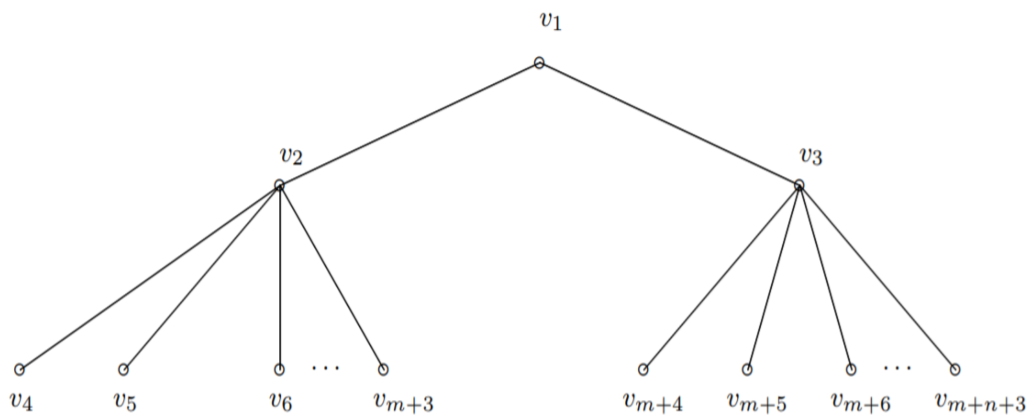

Figure 1 : Graph $\langle K_{1,m}: K_{1,n} \rangle$

Theorem 17 *Let $m, n \geq 2$ be integers. Then the graph $G = \langle K_{1,m}: K_{1,n} \rangle$ admits UISDF with $\gamma_{is}^u(G) = m+n+1$.*

Proof. Consider the graph $G = \langle K_{1,m}: K_{1,n} \rangle$ with vertex set $\{v_1, v_2, \ldots, v_{m+n+3}\}$ as given in Figure. 1. Let f be an UISDF of G. By the definition of UISDF of G, there exists exactly one vertex $v \in V(G)$ such that $f(N[v]) = 1$. By Remark 15, $f(v_i) = 1$ for all $i \neq 1$. Thus $f(N[v_1]) = 1$ and so $f(v_1) = -1$. Therefore $w(f) = m+n+1$ and so $\gamma_{is}^u(G) \geq$

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$m+n+1$.

Let us define a function $f: V(G) \rightarrow \{-1,+1\}$ by

$$f(v_i)=\begin{cases}-1 & when\ i=1\\ +1 & otherwise.\end{cases}$$

From the above labeling, we observe that f is UISDF and $w(f)=m+n+1$. Thus $\gamma_{is}^{u}(G) \leq m+n+1$.

Theorem 18. *For any integer $k \geq 3$, there exists a graph G such that $\gamma_s(G)=\gamma_{is}(G)=\gamma_{is}^{u}(G)=k$.*

Proof. **Case 1:** Suppose k is odd. Then $k=2m+1$ for some $m \geq 1$. Let $G=mK_2 \cup K_1$ be a graph such that $V(G)=\{v_1^i, v_2^i: 1 \leq i \leq m\} \cup \{v_0\}$ and $E(G)=\{v_1^i v_2^i: 1 \leq i \leq m\}$. Let f be an UISDF of G. By Remark 15, $f(v_1^i)=f(v_2^i)=1$. Since $f(N[v_0])=1$, we have $f(v_0)=1$, which implies that $\gamma_{is}^{u}(G) \geq 2m+1=k$. But always $\gamma_{is}^{u}(G) \leq 2m+1=k$. From Lemma 1, we get $k \leq \gamma_s(G) \leq \gamma_{is}(G) \leq \gamma_{is}^{u}(G) \leq k$ and hence $\gamma_s(G)=\gamma_{is}(G)=\gamma_{is}^{u}(G)=k$.

Case 2: Suppose k is even. Then $k=2m$ for some $m \geq 2$.

Let $G=(m-2)K_2 \cup P_3 \cup K_1$ be a graph such that $V(G)=\{v_1^i, v_2^i: 1 \leq i \leq m-2\} \cup \{v_j: 1 \leq j \leq 3\} \cup \{v_0\}$ and $E(G)=\{v_1^i v_2^i: 1 \leq i \leq m-2\} \cup \{v_1 v_{v_2}, v_2 v_3\}$. Let f be an UISDF of G. By Remark 15, $f(v_1^i)=f(v_2^i)=1$ for all i with $1 \leq i \leq m-2$ and $f(v_j)=1$ for $j=1,2,3$. Since $f(N[v_0])=1$, we have $f(v_0)=1$. Thus $\gamma_{is}^{u}(G) \geq 2m=k$. But always $\gamma_{is}^{u}(G) \leq 2m=k$. From Lemma 1, we get $k \leq \gamma_s(G) \leq \gamma_{is}(G) \leq \gamma_{is}^{u}(G) \leq k$ and hence $\gamma_s(G)=\gamma_{is}(G)=\gamma_{is}^{u}(G)=k$.

We say a connected graph H as Category 1 if $\gamma_s(H)=|V(H)|$.

Theorem 19 *Let G be a graph of order n. Then $\gamma_{is}(G)=n$ if and only if $G=\bigcup_{H \in B} H \cup K_1$, where B is the union of some graphs from category 1.*

Proof. Let f be an UISDF. Suppose $G=\bigcup_{H \in B} H \cup K_1$, where B is an union of some graphs from category 1. Then we must have $f(v)=+1$ for all $v \in V(G)$(Since for each $H \in B$, $|V(H)|=\gamma_s(H) \leq \gamma_{is}^{u}(H) \leq |V(H)|$).

Conversely, suppose $G=\bigcup_{H \in B} H \cup K_1$, where B is an union of some graphs from category 1. Let f be an UISDF. Let H be any component of G other than K_1. Suppose $f(u)=-1$ for some $u \in V(H)$. Then $\gamma_s(H) \leq \gamma_{is}^{u}(H) \leq |V(H)|-2$, a contradiction. Thus $f(u)=+1$ for $u \in V(G)$ and hence $w(f)=|V(G)|=n$.

In Theorem 18, it is prove that for integer $k \geq 3$, there exists a disconnected graph G with $\gamma_s(G)=\gamma_{is}(G)=\gamma_{is}^{u}(G)=k$ whereas in the next result we prove that there exists a connected graph G with $\gamma_s(G)=\gamma_{is}(G)=\gamma_{is}^{u}(G)=k$.

Consider the following graphs G_1, G_2 and G_3.

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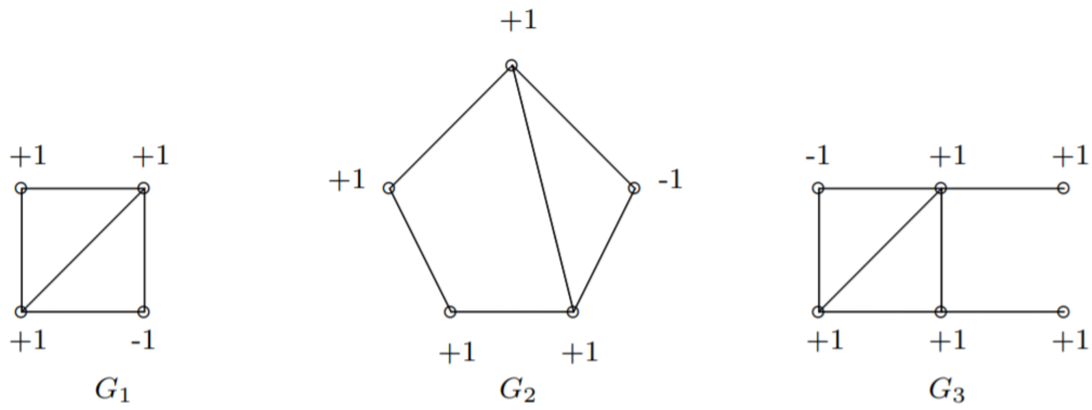

Figure 2 : Graphs G_1, G_2 and G_3.

From the following table :1, it is concluded that the parameter $\gamma_{is}^{u}(G_i) = \gamma_s(G_i) = \gamma_{is}(G_i) = i + 1$.

Table : 1 : Values of $\gamma_{is}^{u}(G_i), \gamma_s(G_i)$ & $\gamma_{is}(G_i)$

Parameter	Graphs		
	G_1	G_2	G_3
γ_s	2	3	4
γ_{is}	2	3	4
γ_{is}^{u}	2	3	4

Theorem 20. *For any integer $k \geq 5$, there exists a connected graph G such that $\gamma_s(G) = \gamma_{is}(G) = \gamma_{is}^{u}(G) = k$.*

Proof. Let $k \geq 5$. Consider the graph $G = \langle K_{1,m}: K_{1,n} \rangle$ with vertex set $\{v_1, v_2, \dots, v_{m+n+3}\}$ as given in Figure. 1, m and n such that $m, n \geq 2$ and then we can choose $m + n + 3 = k + 2$. From Theorem 18, we can have $\gamma_{is}^{u}(G) = k$.

Let f be an ISDF(or SDF) of G. Since the vertices $v_i, 4 \leq i \leq m + n + 3$ are pendent vertices, $f(v_i) = +1$ (otherwise $f(N[v_i]) \leq 0$, a contradiction).

Since the vertices v_2 and v_3 are stem, $f(v_2) = +1$ and $f(v_3) = +1$ (otherwise $f(N[v_4]) \leq 0$ or $f(N[v_{m+n+1}]) \leq 0$, a contradiction).

In this case, $\gamma_{is}(G) \geq k$ and $\gamma_s(G) \geq k$.

Let us define a function $f: V(G) \to \{-1, +1\}$ by

$$f(v_i) = \begin{cases} -1 & when\ i = 1 \\ +1 & otherwise. \end{cases}$$

From the above labeling, we observe that f is ISDF(SDF) with weight $w(f) = m + n + 1 = k$. Thus $\gamma_s(G) \leq k$ and $\gamma_{is}^{u}(G) \leq k$.

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