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Special Graphs of Euler's Family* and Tracing Algorithm- (A New Approach)

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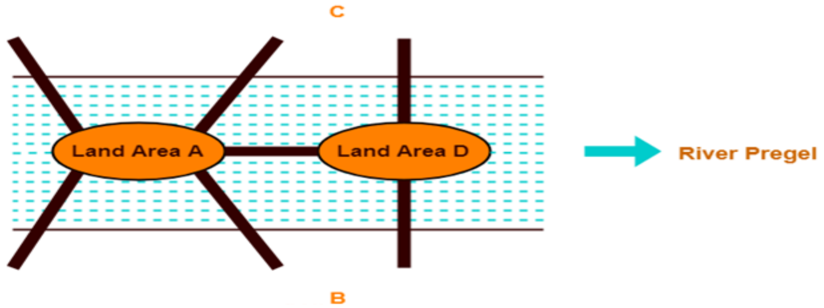
Abstract:
There are in graph theory, some known graphs which date back from centuries. [Euler graph, Hamiltonian graph etc.] These graphs are basic roots for development of graph theory. In this paper we have discussed the novel concept of tracing Euler tour. It depends on the concept of Link vertex - a join vertex of finite number of cycles as components of Euler graph. In addition to this, a new notion of isomorphic transformation of given graph on to a given line segment known as 'Linear Graph' also plays an important role for tracing the Euler graph.
Keywords: Odd edge, even edge, Link Vertex, Linear Graph, Line Graph, Mohmad scimitar, Devil's Star

Abbreviations: L(v), G(L)

Assumption:
The graph under discussion are undirected finite graphs

Introduction:
Graph theory was born in 1736 with Euler's famous paper in which he solved the Konigsberg problem. Euler posed a more general problem in which for a given Euler graph G it is possible to find a closed walk running through every edge of G exactly once. A closed trail, containing all the edges is called an Euler trail. A closed connected graph having an Euler trail is called an Euler graph. Obviously in Euler graph, for every pair of points u and v there exist at least two edge disjoint u-v trails and consequently there are at least two edge –disjoint a u-v paths available to trace an Euler Path. There are some known algorithms for tracing like Fleury's algorithm & Hierholzer's algorithm.
In this paper we have designed our own algorithm – J.P. algorithm, which works very efficiently.

It is well known that Euler confronted with real life problem – seven bridges problem on the river Konigsberg, currently, a city in Russia. There were seven bridges on the river and it was referred to Euler that weather it is possible to traverse on all the seven bridges exactly once and reach the point of origin. Euler came out and successfully responding to the question. The very solution to this point was origin of the graph theory. The figure below shows the real physical situation and the corresponding graphical presentation of the problem on the river Pregel.
Euler was confronted by commuters crossing different bridges to specific land areas of the banks and islands. The question was to suggest a path that initiates from a designated land area traverses on each bridge exactly once in one direction only and return the original point.



[Figure 1: Konigsberg Bridge Problem]

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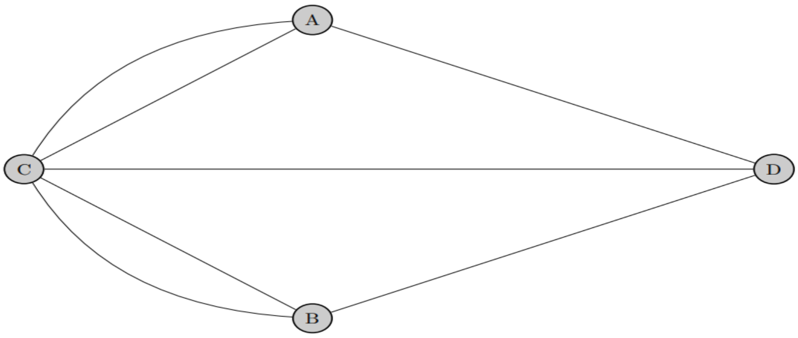
Konigsberg Bridge Problem may be stated as-
"Starting from any of the four land areas A, B, C, or D, is it possible to cross each of the seven bridges exactly once and come back to the point of origin."

Konigsberg Bridge Problem Solution

A Swiss Mathematician Leon hard Euler solved this problem. He observed that traversing on all edges and making a closed tour is not possible.



[Figure 2: Graphical Presentation]

[The points B and C shows land area, and A and D shows island of the river. Line segments show the different seven bridges.]

Euler found that only those networks are traversable in which all the vertices are of the even degree.

Euler graph and fundamental properties:

It is well known that graph theory and some areas of operation research are interrelated; the first one makes clear representation and to some extent enhances the concepts remaining within its peripherals while the second one makes analysis that leads to towards classical work and generalization. Euler's seven bridges problem—origin of graph theory. and Hamiltonian circuit have been continued to remain important focus to the researchers being application oriented. There are a few algorithms to trace the Euler tour but involve more alternatives and procedures. These are descriptive and not time- efficient.

In this note we have lined up with an algorithm that is relatively time efficient and suggests different alternatives to tracing of what is known as Euler's tour/line/ circuit.

The connected graph G is Euler if it is the union of edge disjoint cycles. If $e \in G$ is an edge then e belongs to exactly one cycle c_i ; $i = 1$ to n

Derivation:

Before we discuss some rare old graphs, we consider discuss basics of Euler graph that allow tools to check 'Eulerity' of the graph; these are as follows.

A connected graph G is a Eulerian one if it has a Euler tour- a closed circuit that contains each edge traversed exactly once in one direction only.

This property is identified by checking evenness of degrees of each vertex.

1. **Union of finite cycles:**

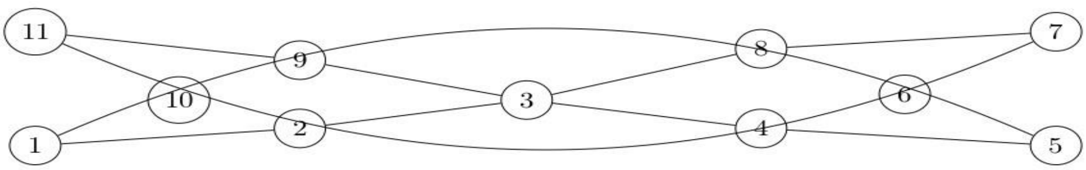
A graph which has either one cycle of even degree vertices or a union of all cycles of even degree vertices is an Euler graph. Euler graph $G = C_1 \cup C_2 \cup \ldots \ldots . . \cup C_n$ where each C_i is an even cycle.

i.e., each cycle has vertices of even degree.

Euler type graphs

We have collection of some rare graphs which observe the above property.

a. Mohmad scimitar - the shape of the graph resembles to curved swords. The old-fashioned Arabian sword.



[Figure 3: Mohmad scimitar]

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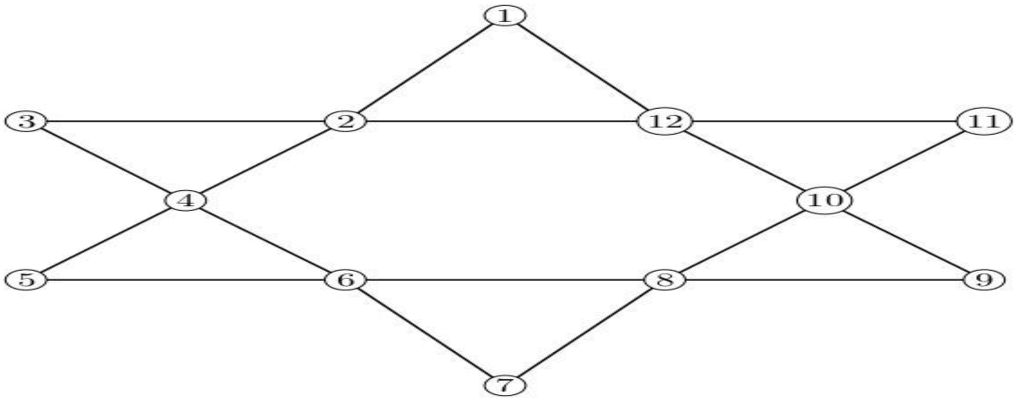
[This graph inherits all the characteristics of Euler graph]

The next graph, known as Devil's Star, also inherits all the characteristics of Euler graph.

It is on the national flag of the country Israel.it is a graph with two equilateral triangles, one with upside down.



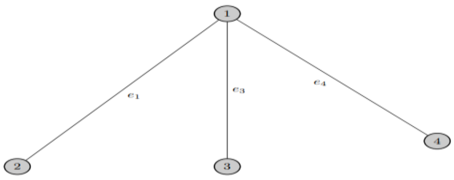
[Figure 4: Devil's Star]

These two graphs are Euler graphs. We will briefly discuss their tracing.

Odd – vertices graph

If all the vertices of a graph are of odd degree than the graph will be noted as odd vertices graph.

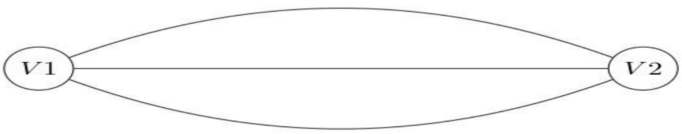
It is obvious that such a graph has even number of vertices.



[Figure 5: Odd Vertices Graph]

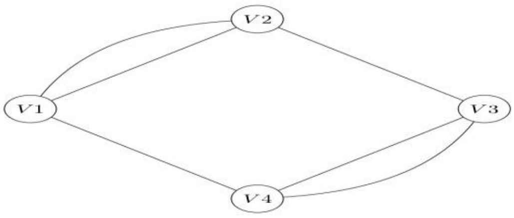
e. g. An Odd – vertices connected graph having

1. Minimal Odd graph – two vertices v_1 and v_2 such that $d(v_1) = d(v_2) = 3$



[Figure 6: Minimal Odd Graph]

2. The next one in this sequence is a connected graph is on 4 vertices v_1, v_2, v_3 and v_4 such that $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 3$



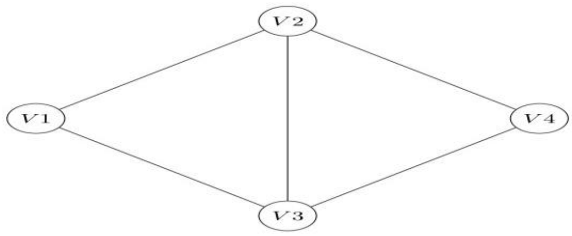
[Figure 7: Connected Graph]

[Comment: By adding an edge only once between two alternate vertices, say joining vertices v_3 and v_4 by an edge, the graph becomes an Euler graph]

a. Unicursal Graphs:

Now we take up graphs which are closer to Euler graph. Such graphs, in general, are called as ‘Unicursal Graphs’ or ‘minimal Euler’.

A graph G is called Unicursal if, in general, even number of its vertices are of odd degree greater than 1. Unicursal graphs are minimal Euler and can be converted to Euler graph by introducing an edge between a pair of two odd vertices. Degree of Unusuality is the number of pairs of odd vertices, It is denoted as $d(G(u))$.



[**Figure 8: First Minimal Connected Unicursal Graph]**

The first minimal connected Unicursal graph such that $D(v_2) = D(v_3) = 3$

[Comment: By joining the vertices v_2 and v_3 by an edge, the graph becomes Euler.]

There are many Unicursal graphs which need special attention. All the properties that indicate Euler graph do fit in this.

[Each vertex is of even degree or it is an union of even cycles.]

Linear Graph of a Graph:

The basic notion of linear graph of a given graph is very important in tracing the Euler tour. Before tracing a Euler graph, it is very essential to verify Eulerity – Evenness of all the vertices.

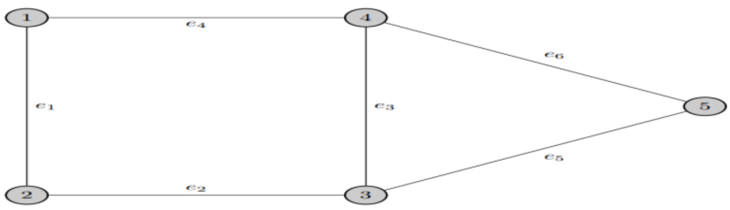
Definition: Linear Graph

If the vertices of the given finite graph be plotted linearly, in ascending order, on a line segment and be joined by arcs, showing edges, the graph so obtained is called a ‘Linear Graph’.

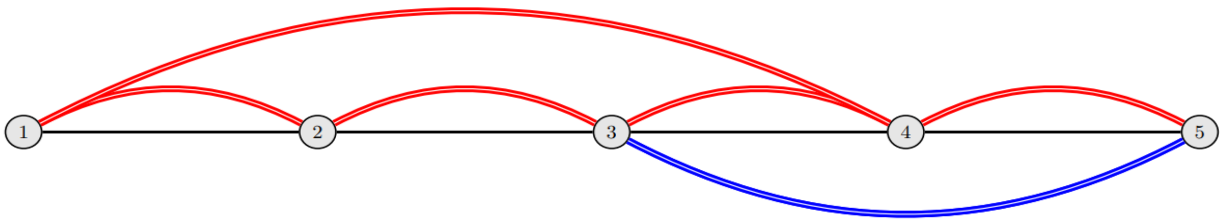
{Note: Number of vertices on the given graph is equal to the number of integer points on the linear graph.}

The number of edges of the given graph is equal to the number of arcs of the linear graph. Also, the edges joining the vertices and the arcs joining the number points (Vertices) have one to one correspondence.

[Consider the following graph G with 5 vertices.]



- **[Figure 9: Graph (G)]**



[Figure 10: Linear Graph G(L)]

Original graph and its corresponding linear graph preserve isomorphic property between the edges and vertices.

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We denote such graphical isomorphism as $G \cong G(L)$.

Properties of vertices correspondence in the graph G is inherited in corresponding linear graph $G(L)$ and the very fact helps us further work.

Some Technical Terms:

In this section we introduce some technical terms which will be useful for further analysis.

1. **An odd Edge:**

We define an odd edge of the graph G as follows.

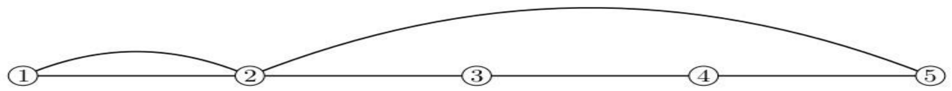
An edge joining any two vertices, say v_i and v_j where $i \in N$ and $j \in N, i \neq j$ $[for\ i = j\ ; it\ is\ a\ self\ loop]$

If $j > i$, then it is known as an odd edge if $j - i$ is odd. For $j > i$ the edge joining vertices v_i and v_j will be known an 'odd edge' if $j - i$ is an odd integer [Equivalently $(i + j)$ is an odd integer]

In the same way, if $j - i$ is an even integer then the edge joining vertices v_i and v_j will be known an 'even edge' [or $(j + i)$is an even integer]

We follow a tradition; the odd edge is shown above the line segment and the even edge is shown below the line segment.

Let G* be a linear graph of the given graph G with the vertices shown bellow



[Figure 11: G(L) with Odd Edges]

The edge/ arcs joining vertex 1 with vertex 2, and verte2 with vertex 5, i.e. edges 1-2 and 2-5 are odd edges.

[Note: Isomorphism between the vertices and edges between the given graph G and the line graph $G(L)$. The definition holds true in both cases.]

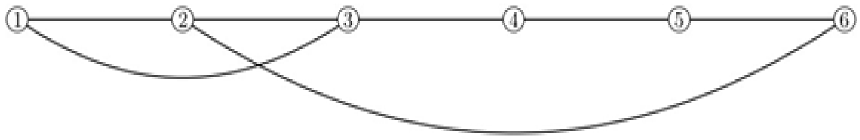
2. **Even Edge:**

We define even edge of the graph G as follows.

The edge joining the vertices v_i and v_j, any two distinct vertices of the graph G with $i < j$ being positive integers, is known as an even edge if $(j - i)$is an even integer. [Equivalently, $(j + i)$is an even integer.]

In the following linear graph, the edge 1-3 joining the vertex 1 with the vertex 3 is an even edge.;

as 3-1 = 2 –An even integer. [or the sum 3 + 1 = 4 is even.] In the same way the edge 2-6, joining the vertices 2 and 6 is an even edge



[Figure 12: G(L) with Even Edges]

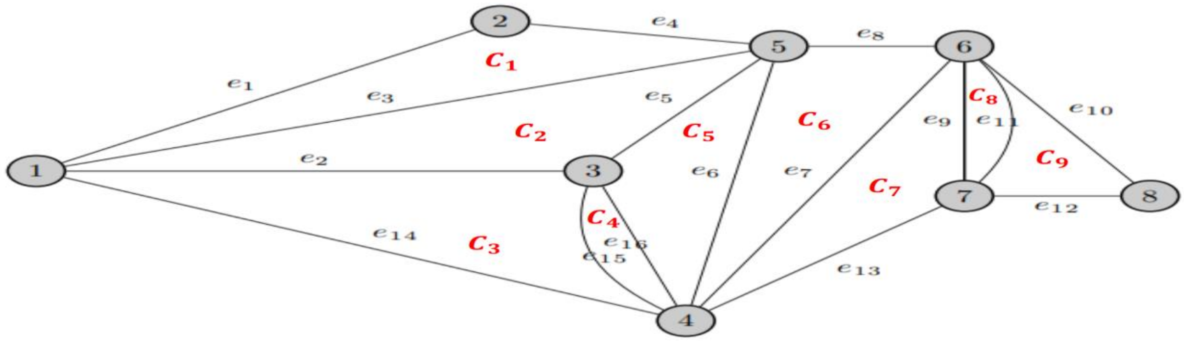
[Note: Isomorphism between the vertices and edges between the given graph G and the line graph G(L). The definition holds true in both cases.]

5.4 A link Vertex:

We now introduce an important and useful notion for the further development of the work that we aim at.

A vertex V_L in the graph G is said to be a link vertex if it is a common vertex to at least two different cycles having all vertices of even degree. It can be considered as initial and terminus vertex of each cycle. Also, we note that from all given cycles of even degree no cycle has a common edge.

Also, it is important to note that there can be more than one link vertices in a given graph.

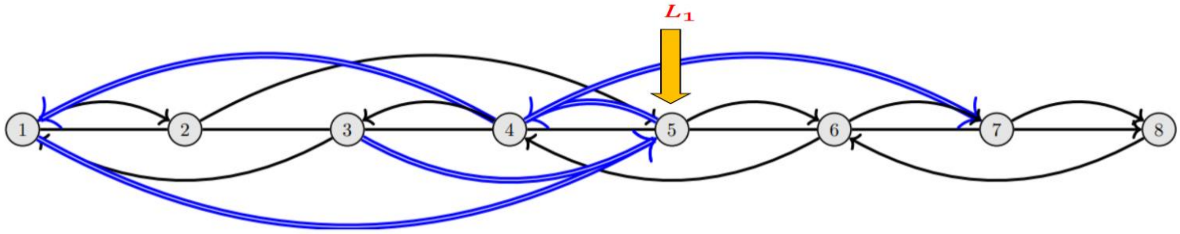


[Figure 13: $G = C_1 \cup C_2 \cup \dots\dots..\cup C_9$]

Cycle C_1 joining the vertices 1-2-5-1. Cycle C_2 joining the vertices 1-3-5-1. etc.…

The given Euler graph $G = C_1 \cup C_2 \cup \dots\dots..\cup C_9$

L_1 is a link vertex (vertex 5 no.) between C_1 and C_2 also between C_1, C_2, C_5 and C_6



[Figure 14: Linear Graph of G]

Linear graph of Graph G with vertex 5 is a link vertex.

Properties of Link Vertex:

The salient features of the link vertex are very important as they help identify the different cycles.

(1) it is very important to note that the degree of the link vertex is at least 4 or an even integer greater than 4. If the vertex L is a link vertex, then $d(L) \geq 2n;\ n \in N - \{1\}$.

(2) Two or more different cycles may have only one link vertex.

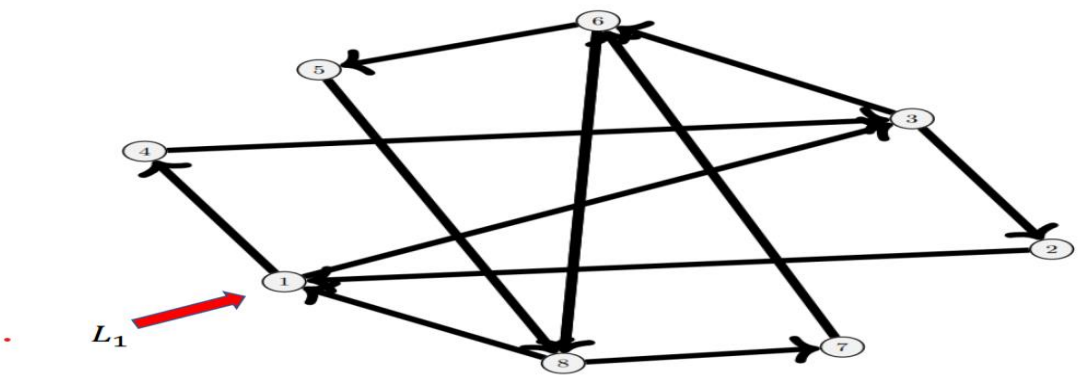
(3) Link vertex of a graph and its corresponding linear graph is same.

(4) A graph which has either one cycle of even degree or it has at least one link vertex is a Euler graph. $G = C_1 \cup C_2$

6 Tracing of Euler graph—A new algorithm

There are some known algorithms like Fleury's algorithm & Hierholzer's algorithm. Which help construct/trace Euler tour but are relatively more descriptive than what we have suggested.

We find that all the time tracing of Euler graph is not simple following known algorithms. Concept of using link vertices for tracing will make it simple. As an example, we have Euler graph



[Figure 15: Tracing of Graph through link vertex]

J – P Algorithm

In this unit, we discuss J.P. algorithm and understand our simple logic for tracing Euler graph/tour.

We have already discussed the notion of link vertex and the same is very useful in establishing the main objective of this paper.

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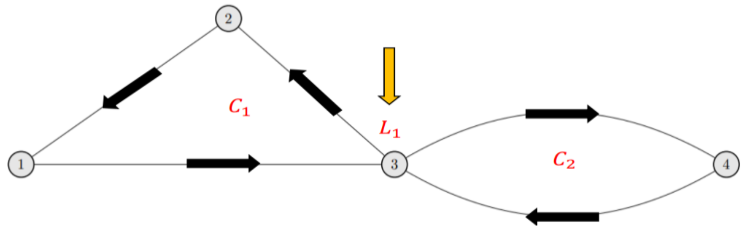
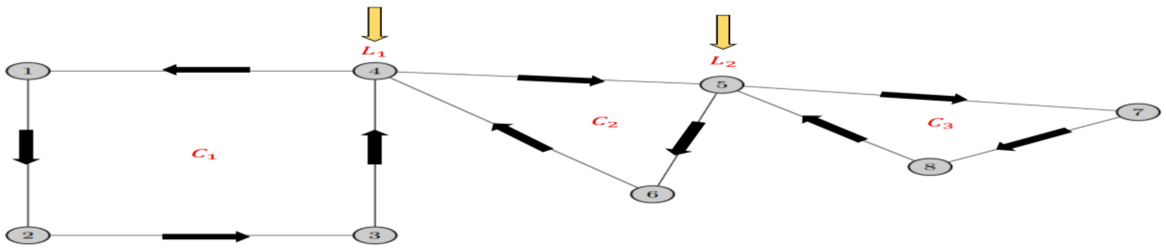
1. Let G be Euler graph we know that G can be decomposed of finite number of n cycles $(n \geq 1)$
2.
a. If $n = 1$, the graph itself is easily traced.
b. Let $G = C_1 \cup C_2 \cup C_3 \cup \dots \dots \dots \cup C_n$
By definition of link vertex. There are '$n - 1$' link vertices connecting different cycles. Let these be denoted as $L_1, L_2, \dots \dots \dots, L_{n-1}$
3. Now we focus on tracing procedure.
a. Begin from L_1 and trace C_1
b. Once you return on L_1 (on finishing tracing of C_1) trace C_2 from L_1 and you will end up in L_1.this is an Euler graph with two cycles C_1 $and C_2$ with L_1 is a link vertex.
4. If the graph is not completely traced then we have $G = C_1 \cup C_2 \cup C_3$; where link vertices $L_1 = C_1 \cap C_2$ and $L_2 = C_2 \cap C_3$. Now moving on C_2, once we reach L_2,complete tracing of C_3 and come back of L_1 passing through L_2.
5. Continue tracing of all possible cycles through successively passing through $L_1, L_2, \dots \dots \dots$

Illustration 1.
Graph $G_1 = C_1 \cup C_2$ with link vertex $L_1 = C_1 \cap C_2$



[Figure 16: $G_1 = C_1 \cup C_2$]
$G_1 = C_1 \cup C_2$
[As shown above, may begin with L_1, trace C_1 come back to L_1 and then tracing C_2 and come back to $\boldsymbol{L_1}$.**]**
Illustration 2.
The Euler graph $G_1 = C_1 \cup C_2 \cup C_3$
with link vertex $L_1 = C_1 \cap C_2, L_2 = C_2 \cap C_3$



[Figure 17: Euler graph $G_1 = C_1 \cup C_2 \cup C_3$]
[Tracing begins from L_1 completes C_1 and returns L_1.
Now it moves on C_2 to reach the next link vertex L_2.
Now trace C_3 and come back to L_2 and completing tour on C_2
Come back to L_1. This is the complete tracing.
[Note: It is initially very important to identify the cycles – cycle decomposition and then link vertices]

6.1 Tracing with linear graph
Some key points for tracing Euler tour:
In this important unit we point out salient features for tracing Euler tour.
1. If the given graph is Euler one then as a rule of thumb, one can trace it from any vertex and move one edges as the edge advances towards the link vertex.
2. Experience claims that tracing from one and towards remaining link vertices is a one-time master – stroke.

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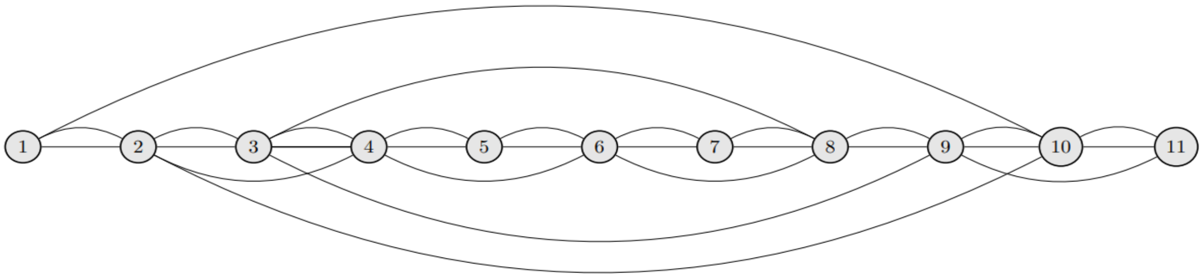
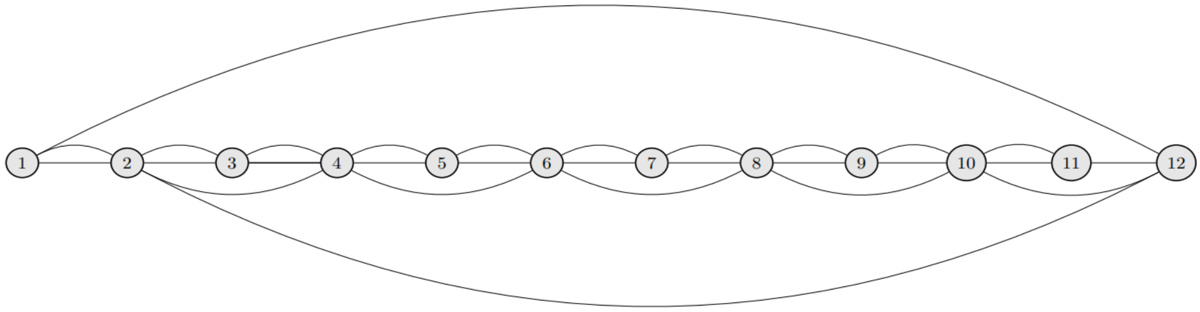


[Figure 18: Tracing of Mohmad scimitar]



[Figure 19: Tracing of Devil's Star]

Discussion:

In this paper we discuss about the algorithm to construct a Euler Path in Euler Graph by using link vertex. For that here we study algorithm i.e., Fleury's & Hierholzer's. Both are very important algorithm for determination of Euler path in Euler graph. Also, both algorithms are complicated. So, we derived J – P algorithm, that is the easiest algorithm for determination of Euler path in Euler graph.

Conclusion: Tracing of Euler tour, above all known previous algorithm, has been simplified using 'Link Vertex' and linear graphs a new concept is simplified.

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