The Ciência & Engenharia - Science & Engineering Journal
ISSN: 0103-944X
Volume 11 Issue 1, 2023
pp: 2195 - 2205

Union Of 4-Total Prime Cordial Graph G With The Bistar $B_{n,N}$

J. Maruthamani[1] And S. Pasunkili Pandian[2]

Abstract. Let G be a (p, q) graph. Let $f : V(G) \to \{1, 2, \ldots, k\}$ be a map where $k \in \mathbb{N}$ is a variable and $k > 1$. For each edge uv, assign the label $\gcd(f(u), f(v))$. The map f is called a k- Total prime cordial labeling of G if $|t_{pf}(i) - t_{pf}(j)| \leq 1$, $i, j \in \{1, 2, \cdots, k\}$ where $t_{pf}(x)$ denotes the total number of vertices and the edges labelled with x. A graph with a k-total prime cordial labeling is called k-total prime cordial graph. In this paper we investigate the 4-total prime cordial labeling of $G \cup B_{n,n}$, where G has a 4-total prime cordial labeling and $B_{n,n}$ is a bistar.

1. Introduction

Graphs considered here are finite, simple and undirected. A weaker version of graceful and harmonious labeling called cordial labeling was introduced by cahit [2]. Several cordial related labelings have been studied in [1, 10, 4]. Ponraj et al. [6], have been introduced the notion of k-total prime cordial labeling and the k-total prime cordial labeling of some graphs have been investigated. In this paper, we investigate the 4-total prime cordial labeling of $G \cup B_{n,n}$, where G has a 4-total prime cordial labeling and $B_{n,n}$ is a bistar.

2. *k*-total prime cordial labeling

Definition 2.1. Let G be a (p, q) graph. Let $f : V(G) \to \{1, 2, \ldots, k\}$ be a function where $k \in \mathbb{N}$ is a variable and $k > 1$. For each edge uv, assign the label $\gcd(f(u), f(v))$. The map f is called k-Total prime cordial labeling of G if $|t_{pf}(i) - t_{pf}(j)| \leq 1$, $i, j \in \{1, 2, \cdots, k\}$ where $t_{pf}(x)$ denotes the total number of vertices and the edges labelled with x. A graph with a k-total prime cordial labeling is called k-total prime cordial graph.

2000 *Mathematics Subject Classification.* 05C78.

Key words and phrases. Prime Cordial Graph, k-total prime cordial graph, Bis- tar, Union of Graphs.

Definition 2.2. The *union* of two graphs G_1 and G_2 is the graph $G_1 \cup G_2$ with $V(G_1 \cup G_2) = V(G_1) \cup V(G_2)$ and $E(G_1 \cup G_2) = E(G_1) \cup E(G_2)$.

Definition 2.3. A *bipartite graph* is a graph whose vertex set $V(G)$ can be partitioned into two subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2. If G contains every edge joining V_1 and V_2, then G is a *complete bipartite graph*. If $|V_1| = m$ and $|V_2| = n$, then the complete bipartite graph is denoted by $K_{m,n}$.

Definition 2.4. $K_{1,n}$ is called a *star*.

Definition 2.5. The *bistar* $B_{m,n}$ is the graph obtained by making adjacent the two central vertices of $K_{1,m}$ and $K_{1,n}$.

3. Preliminaries

Theorem 3.1. [6] The bistar $B_{n,n}$ is 4-total prime cordial for all values of n.

Before proving the main theorem, we once again defined the 4-total prime cordial labeling g of the bistar $B_{n,n}$:

For $n = 4k$ and $k \in \mathbb{N}$. The labeling pattern is given in Table 1

Vertices	Labels
u	4
v	3
$u_1, \ldots, u_{2k}$	4
$u_{2k+1}, \ldots, u_{4k}$	2
$v_1, \ldots, v_{2k}$	3
$v_{2k+1}, \ldots, v_{4k}$	1

Table 1:

In the case of $n = 4k + 1$ and $k \in \mathbb{N}$. The labeling pattern is given in Table 2

The Ciência & Engenharia - Science & Engineering Journal
ISSN: 0103-944X
Volume 11 Issue 1, 2023
pp: 2195 - 2205

Vertices	Labels
u	4
v	3
$u_1, \dots, u_{2k}$	4
$u_{2k+1}, \dots, u_{4k}$	2
u_{4k+1}	2
$v_1, \dots, v_{2k}$	3
$v_{2k+1}, \dots, v_{4k}$	1
v_{4k+1}	4

Table 2:

For $n = 4k + 2$ and $k \in \mathbb{N}$. The labeling patter n is given in Table 3

Vertices	Labels
u	4
v	3
$u_1, \dots, u_{2k}$	4
$u_{2k+1}, \dots, u_{4k}$	2
u_{4k+1}	2
u_{4k+2}	1
$v_1, \dots, v_{2k}$	3
$v_{2k+1}, \dots, v_{4k-1}$	1
v_{4k}	3
v_{4k+1}	2
v_{4k+2}	4

n is given in Table 3
In the case of $n = 4k + 3$ and $k \in \mathbb{N}$. The labeling pattern is given in Table 4

Vertices	Labels
u	4
v	3
$u_1, \dots, u_{2k}$	4
$u_{2k+1}, \dots, u_{4k}$	2
u_{4k+1}	2
u_{4k+2}	1
u_{4k+3}	4
$v_1, \dots, v_{2k}$	3
$v_{2k+1}, \dots, v_{4k-1}$	1
v_{4k}	3
v_{4k+1}	2
v_{4k+2}	4
v_{4k+3}	2

The Ciência & Engenharia - Science & Engineering Journal
ISSN: 0103-944X
Volume 11 Issue 1, 2023
pp: 2195 - 2205

Table 4: For $n \in \{1, 2, 3\}$. The labeling pattern is given in Table 5

n	u	v	u_1	u_2	u_3	v_1	v_2	v_3
$B_{1,1}$	2	3	4			4		
$B_{2,2}$	4	3	4	2		3	1	
$B_{3,3}$	4	3	4	2	2	3	1	4

Table 5:

Remark. 2-*total prime cordial graph is* 2-*total product cordial graph.*

4. Main Results

Theorem 4.1. Let G be a (p,q) 4- total prime cordial graph then, $G \cup B_{n,n}$ is 4- total prime cordial for all $n \geq 4$.

Proof. Let u, v be the central vertices of the bistar $B_{n,n}$ and u_i $(1 \leq i \leq n)$ be the pendent vertices adjacent to u and v_i $(1 \leq i \leq n)$ be the pendent vertices adjacent to v. Let f be the 4- total prime cordial labeling of G and g be the 4- total prime cordial labeling of bistar $B_{n,n}$ as in Theorem 3.1. We now define $\varphi : V(G) \cup V(B_{n,n}) \to \{1, 2, 3, 4\}$ by

$$\varphi(u) = \begin{cases} f(u), & if\ \ u \in V(G) \\ g(u), & if\ \ u \in B_{n,n} \end{cases}.$$

To prove our result, we have to split the proof into 15 cases.

Case 1. $p + q \equiv 0 \pmod 4$.

Suppose $t_{pf}(1) = t_{pf}(2) = t_{pf}(3) = t_{pf}(4) = r$.

Subcase 1(a). $n \equiv 0 \pmod 4$.

Let $n = 4k$ and $k \in \mathbb{N}$. As in case 1 of Theorem 3.1, fix the label to the vertices u, v, u_i $(1 \leq i \leq 4k)$ and v_i $(1 \leq i \leq 4k)$ together with the 4- total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = 4k + r + 1$ and $t_{p\varphi}(4) = 4k + r$.

Subcase 1(b). $n \equiv 1 \pmod 4$.

Let $n = 4k + 1$ and $k \in \mathbb{N}$. As in case 2 of Theorem 3.1, fix the label to the vertices u, v, u_i $(1 \leq i \leq 4k + 1)$ and v_i $(1 \leq i \leq 4k + 1)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(4) = 4k + r + 2$ and $t_{p\varphi}(3) = 4k + r + 1$.

Subcase 1(c). $n \equiv 2 \pmod 4$.

Let $n = 4k + 2$ and $k \in \mathbb{N}$. As in case 3 of Theorem 3.1, fix the label to the vertices u, v, u_i $(1 \leq i \leq 4k + 2)$ and v_i $(1 \leq i \leq 4k + 2)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = 4k + r + 3$ and $t_{p\varphi}(4) = 4k + r + 2$.

Subcase 1(d). $n \equiv 3 \pmod 4$.

Let $n = 4k + 3$ and $k \in \mathbb{N}$. As in case 4 of Theorem 3.1, fix the label to the vertices u, v, u_i $(1 \leq i \leq 4k + 3)$ and v_i $(1 \leq i \leq 4k + 3)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(4) = 4k + r + 4$ and $t_{p\varphi}(3) = 4k + r + 3$.

Case 2. $p + q \equiv 1 \pmod 4$.

Suppose $t_{pf}(1) = t_{pf}(2) = t_{pf}(3) = r$ and $t_{pf}(4) = r + 1$.

Subcase 2(a). $n \equiv 0 \pmod 4$.

Let $n = 4k$ and $k \in \mathbb{N}$. As in case 1 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \leq i \leq 4k)$ and v_i $(1 \leq i \leq 4k)$. Next we relabel the vertices 3, 4 and 2 by u_{2k}, v_{4k-1} and v_{4k} respectively. Let h be this relabeled technique of the bistar $B_{n,n}$. Let

$$\varphi(u) = \begin{cases} f(u), & if\ \ u \in V(G) \\ g(u), & if\ \ u \in B_{n,n} \end{cases} \ldots\ldots\ldots\ldots\ldots(1).$$

Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 1$.

Subcase 2(b). $n \equiv 1 \pmod 4$.

The Ciência & Engenharia - Science & Engineering Journal
ISSN: 0103-944X
Volume 11 Issue 1, 2023
pp: 2195 - 2205

Let $n = 4k + 1$ and $k \in \mathbb{N}$. As in case 2 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 1)$ and v_i $(1 \le i \le 4k + 1)$. Finally we relabel the vertex u_{2k} by 3 and v_{4k} by 4. Let h be defined by the relabeled technique of the bistar $B_{n,n}$ in (1). Let φ be the function as in (1). It is easy to verify that $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 2$.

Subcase 2(c). $n \equiv 2 \pmod 4$.

Let $n = 4k + 2$ and $k \in \mathbb{N}$. As in case 3 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 2)$ and v_i $(1 \le i \le 4k + 2)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 3$.

Subcase 2(d). $n \equiv 3 \pmod 4$.

Let $n = 4k + 3$ and $k \in \mathbb{N}$. As in case 4 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 3)$ and v_i $(1 \le i \le 4k + 3)$. Finally we relabel the vertex u_{4k+2} by 3 and v_{4k+1} by 1. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 4$ where h and φ defined as in (1).

Case 3. $p + q \equiv 1 \pmod 4$.

Suppose $t_{pf}(1) = t_{pf}(2) = t_{pf}(4) = r$ and $t_{pf}(3) = r + 1$.

Subcase 3(a). $n \equiv 0 \pmod 4$.

Let $n = 4k$ and $k \in \mathbb{N}$. As in case 1 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k)$ and v_i $(1 \le i \le 4k - 3)$. Next we assign the labels 2, 2 and 2 to the vertices v_{4k-2}, v_{4k-1} and v_{4k} respectively. Let h be defined by the relabeled technique of the bistar $B_{n,n}$ in (1). Let φ be the function as in (1). It is easy to verify that $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 1$.

Subcase 3(b). $n \equiv 1 \pmod 4$.

Let $n = 4k + 1$ and $k \in \mathbb{N}$. As in case 2 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 1)$ and v_i $(1 \le i \le 4k + 1)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 2$.

Subcase 3(c). $n \equiv 2 \pmod 4$.

Let $n = 4k + 2$ and $k \in \mathbb{N}$. As in case 3 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 2)$ and v_i $(1 \le i \le 4k + 2)$. Finally we relabel the vertex u_{4k+2} by 3 and v_{4k+2} by 4. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 3$ where h and φ defined as in (1).

Subcase 3(d). $n \equiv 3 \pmod 4$.

Let $n = 4k + 3$ and $k \in \mathbb{N}$. As in case 4 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 3)$ and v_i $(1 \le i \le 4k + 3)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 4$.

Case 4. $p + q \equiv 1 \pmod 4$.

Suppose $t_{pf}(1) = t_{pf}(3) = t_{pf}(4) = r$ and $t_{pf}(2) = r + 1$.

Subcase 4(a). $n \equiv 0 \pmod 4$.

Let $n = 4k$ and $k \in \mathbb{N}$. As in case 1 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k)$ and v_i $(1 \le i \le 4k)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 1$.

Subcase 4(b). $n \equiv 1 \pmod 4$.

Let $n = 4k + 1$ and $k \in \mathbb{N}$. As in case 2 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 1)$ and v_i $(1 \le i \le 4k + 1)$. Finally we relabel the vertex u_{4k} by 3 and v_{4k-1} by 2. Let h be defined by the relabeled technique of the bistar $B_{n,n}$ in (1). Let φ be the function as in (1). It is easy to verify that $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 2$.

Subcase 4(c). $n \equiv 2 \pmod 4$.

Let $n = 4k + 2$ and $k \in \mathbb{N}$. As in case 3 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 2)$ and v_i $(1 \le i \le 4k + 2)$. Finally we relabel the vertex v_{4k} by 4. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 3$ where h and φ defined as in (1).

Subcase 4(d). $n \equiv 3 \pmod 4$.

Let $n = 4k + 3$ and $k \in \mathbb{N}$. As in case 4 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 3)$ and v_i $(1 \le i \le 4k + 3)$. Finally we relabel the vertex v_{4k} by 3. Let h be defined by the relabeled technique of the bistar $B_{n,n}$ in (1). Let φ be the function as in (1). It is easy to verify that $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(4) = 4k + r + 4$ and $t_{p\varphi}(3) = 4k + r + 5$.

The Ciência & Engenharia - Science & Engineering Journal
ISSN: 0103-944X
Volume 11 Issue 1, 2023
pp: 2195 - 2205

Case 5. $p + q \equiv 1 \pmod 4$.
Suppose $t_{pf}(2) = t_{pf}(3) = t_{pf}(4) = r$ and $t_{pf}(1) = r + 1$.

Subcase 5(a). $n \equiv 0 \pmod 4$.
Let $n = 4k$ and $k \in \mathbb{N}$. As in case 1 of Theorem 3.1, fix the label to the vertices u, v, u_i $(1 \leq i \leq 4k)$ and v_i $(1 \leq i \leq 4k)$. Finally we relabel the vertex v_{4k} by 2. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k+r+1$ where h and φ defined as in (1).

Subcase 5(b). $n \equiv 1 \pmod 4$.
Let $n = 4k + 1$ and $k \in \mathbb{N}$. As in case 2 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \leq i \leq 4k + 1)$ and v_i $(1 \leq i \leq 4k + 1)$. Finally we relabel the vertex u_{4k+1} by 3, v_{4k-1} and v_{4k} by 2. Let h be defined by the relabeled technique of the bistar $B_{n,n}$ in (1). Let φ be the function as in (1). It is easy to verify that $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 2$.

Subcase 5(c). $n \equiv 2 \pmod 4$.
Let $n = 4k+2$ and $k \in \mathbb{N}$. As in case 3 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \leq i \leq 4k + 2)$ and v_i $(1 \leq i \leq 4k+2)$. Finally we relabel the vertex u_{4k+2} by 4 and v_{4k+1} by 1. Let h be defined by the relabeled technique of the bistar $B_{n,n}$ in (1). Let φ be the function as in (1). It is easy to verify that $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 3$.

Subcase 5(d). $n \equiv 3 \pmod 4$.
Let $n = 4k + 3$ and $k \in \mathbb{N}$. As in case 4 of Theorem 3.1, fix the label to the vertices u, v, u_i $(1 \leq i \leq 4k+3)$ and v_i $(1 \leq i \leq 4k+3)$. Finally we relabel the vertex v_{4k+2} by 3. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 4$ where h and φ defined as in (1).

Case 6. $p + q \equiv 2 \pmod 4$.
Suppose $t_{pf}(1) = t_{pf}(2) = r + 1$ and $t_{pf}(3) = t_{pf}(4) = r$.

Subcase 6(a). $n \equiv 0 \pmod 4$.
Let $n = 4k$ and $k \in \mathbb{N}$. As in case 1 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \leq i \leq 4k)$ and v_i $(1 \leq i \leq 4k)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = 4k + r + 2$ and $t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 1$.

Subcase 6(b). $n \equiv 1 \pmod 4$.
Let $n = 4k + 1$ and $k \in \mathbb{N}$. As in case 2 of Theorem 3.1, fix the label to the vertices u, v, u_i $(1 \leq i \leq 4k + 1)$ and v_i $(1 \leq i \leq 4k + 1)$. Finally we relabel the vertex u_{4k+1} by 3 and v_{4k} by 2. Then $t_{p\varphi}(1) = 4k+r+3$ and $t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 2$ where h and φ defined as in (1).

Subcase 6(c). $n \equiv 2 \pmod 4$.

Let $n = 4k+2$ and $k \in \mathbb{N}$. The proof is similar to Subcase 5(c) in Case 5. Then $t_{p\varphi}(1) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 3$ and $t_{p\varphi}(2) = 4k + r + 4$.

Subcase 6(d). $n \equiv 3 \pmod 4$.

Let $n = 4k+3$ and $k \in \mathbb{N}$. The proof is similar to Subcase 5(d) in Case 5. Then $t_{p\varphi}(1) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 4$ and $t_{p\varphi}(2) = 4k + r + 5$.

Case 7. $p + q \equiv 2 \pmod 4$.
Suppose $t_{pf}(1) = t_{pf}(3) = r + 1$ and $t_{pf}(2) = t_{pf}(4) = r$.

Subcase 7(a). $n \equiv 0 \pmod 4$.
Let $n = 4k$ and $k \in \mathbb{N}$. The proof is similar to Subcase 5(a) in Case 5. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(4) = 4k + r + 1$ and $t_{p\varphi}(3) = 4k + r + 2$.

Subcase 7(b). $n \equiv 1 \pmod 4$.
Let $n = 4k+1$ and $k \in \mathbb{N}$. As in case 2 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \leq i \leq 4k + 1)$ and v_i $(1 \leq i \leq 4k + 1)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = 4k + r + 3$ and $t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 2$.

Subcase 7(c). $n \equiv 2 \pmod 4$.

Let $n = 4k+2$ and $k \in \mathbb{N}$. The proof is similar to Subcase 5(c) in Case 5. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(4) = 4k + r + 3$ and $t_{p\varphi}(3) = 4k + r + 4$.

Subcase 7(d). $n \equiv 3 \pmod 4$.
Let $n = 4k+3$ and $k \in \mathbb{N}$. As in case 4 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \leq i \leq 4k + 3)$ and v_i $(1 \leq i \leq 4k + 3)$ together with the 4-total prime cordial f of G is also satisfies the condition of

The Ciência & Engenharia - Science & Engineering Journal
ISSN: 0103-944X
Volume 11 Issue 1, 2023
pp: 2195 - 2205

4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = 4k + r + 5$ and $t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 4$.

Case 8. $p + q \equiv 2 \pmod 4$.

Suppose $t_{pf}(1) = t_{pf}(4) = r + 1$ and $t_{pf}(2) = t_{pf}(3) = r$.

Subcase 8(a). $n \equiv 0 \pmod 4$.

Let $n = 4k$ and $k \in \mathbb{N}$. The proof is similar to Subcase 5(a) in Case 5. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = 4k + r + 1$ and $t_{p\varphi}(4) = 4k + r + 2$.

Subcase 8(b). $n \equiv 1 \pmod 4$.

Let $n = 4k + 1$ and $k \in \mathbb{N}$. As in case 2 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 1)$ and v_i $(1 \le i \le 4k + 1)$. Finally we relabel the vertex u_{4k} by 3 and u_{4k+1} by 1. Let h be defined by the relabeled technique of the bistar $B_{n,n}$ in (1). Let φ be the function as in (1). It is easy to verify that $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(4) = 4k + r + 2$ and $t_{p\varphi}(3) = 4k + r + 3$.

Subcase 8(c). $n \equiv 2 \pmod 4$.

Let $n = 4k + 2$ and $k \in \mathbb{N}$. As in case 3 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 2)$ and v_i $(1 \le i \le 4k + 2)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = 4k + r + 4$ and $t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 3$.

Subcase 8(d). $n \equiv 3 \pmod 4$.

Let $n = 4k + 3$ and $k \in \mathbb{N}$. The proof is similar to Subcase 5(d) in Case 5. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = 4k + r + 4$ and $t_{p\varphi}(4) = 4k + r + 5$.

Case 9. $p + q \equiv 2 \pmod 4$.

Suppose $t_{pf}(2) = t_{pf}(3) = r + 1$ and $t_{pf}(1) = t_{pf}(2) = r$.

Subcase 9(a). $n \equiv 0 \pmod 4$.

Let $n = 4k$ and $k \in \mathbb{N}$. As in case 1 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k)$ and v_i $(1 \le i \le 4k)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(4) = 4k + r + 1$ and $t_{p\varphi}(3) = 4k + r + 2$.

Subcase 9(b). $n \equiv 1 \pmod 4$.

Let $n = 4k + 1$ and $k \in \mathbb{N}$. As in case 2 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 1)$ and v_i $(1 \le i \le 4k + 1)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 2$ and $t_{p\varphi}(2) = 4k + r + 3$.

Subcase 9(c). $n \equiv 2 \pmod 4$.

Let $n = 4k + 2$ and $k \in \mathbb{N}$. As in case 3 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 2)$ and v_i $(1 \le i \le 4k + 2)$. Finally we relabel the vertex u_{4k+2} by 3 and v_{4k+2} by 4. Let h be defined by the relabeled technique of the bistar $B_{n,n}$ in (1). Let φ be the function as in (1). It is easy to verify that $t_{p\varphi}(1) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 3$ and $t_{p\varphi}(2) = 4k + r + 4$.

Subcase 9(d). $n \equiv 3 \pmod 4$.

Let $n = 4k + 3$ and $k \in \mathbb{N}$. As in case 4 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 3)$ and v_i $(1 \le i \le 4k + 3)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(2) = 4k + r + 5$ and $t_{p\varphi}(1) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 4$.

Case 10. $p + q \equiv 2 \pmod 4$.

Suppose $t_{pf}(2) = t_{pf}(4) = r + 1$ and $t_{pf}(1) = t_{pf}(3) = r$.

Subcase 10(a). $n \equiv 0 \pmod 4$.

Let $n = 4k$ and $k \in \mathbb{N}$. As in case 1 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k)$ and v_i $(1 \le i \le 4k)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = 4k + r + 1$ and $t_{p\varphi}(4) = 4k + r + 2$.

Subcase 10(b). $n \equiv 1 \pmod 4$.

Let $n = 4k + 1$ and $k \in \mathbb{N}$. As in case 2 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 1)$ and v_i $(1 \le i \le 4k + 1)$. Finally we relabel the vertex u_{4k+1} by 3 and v_{4k+1} by 2. Let h be defined by the relabeled technique of the bistar $B_{n,n}$ in (1). Let φ be the function as in (1). It is easy to verify that $t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 2$ and $t_{p\varphi}(1) = 4k + r + 3$.

Subcase 10(c). $n \equiv 2 \pmod 4$.

Let $n = 4k + 2$ and $k \in \mathbb{N}$. As in case 3 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 2)$ and v_i $(1 \le i \le 4k + 2)$ together with the 4-total prime cordial f of G is also satisfies the condition of

The Ciência & Engenharia - Science & Engineering Journal
ISSN: 0103-944X
Volume 11 Issue 1, 2023
pp: 2195 - 2205

4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 3$ and $t_{p\varphi}(2) = 4k + r + 4$.

Subcase 10(d). $n \equiv 3 \pmod 4$.
Let $n = 4k + 3$ and $k \in N$. As in case 4 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 3)$ and v_i $(1 \le i \le 4k + 3)$. Finally we relabel the vertex u_{4k+2} by 3 and v_{4k+3} by 1. Let h be defined by the relabeled technique of the bistar $B_{n,n}$ in (1). Let φ be the function as in (1). It is easy to verify that $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = 4k + r + 4$ and $t_{p\varphi}(4) = 4k + r + 5$.

Case 11. $p + q \equiv 2 \pmod 4$.
Suppose $t_{pf}(3) = t_{pf}(4) = r + 1$ and $t_{pf}(1) = t_{pf}(2) = r$.

Subcase 11(a). $n \equiv 0 \pmod 4$.
Let $n = 4k$ and $k \in N$. As in case 1 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k)$ and v_i $(1 \le i \le 4k)$. Finally we relabel the vertex u_{2k} by 1, v_{4k-1} by 4 and v_{4k} by 2. Let h be defined by the relabeled technique of the bistar $B_{n,n}$ in (1). Let φ be the function as in (1). It is easy to verify that $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(4) = 4k + r + 1$ and $t_{p\varphi}(3) = 4k + r + 2$.

Subcase 11(b). $n \equiv 1 \pmod 4$.
Let $n = 4k + 1$ and $k \in N$. As in case 2 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 1)$ and v_i $(1 \le i \le 4k + 1)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = 4k + r + 2$ and $t_{p\varphi}(4) = 4k + r + 3$.

Subcase 11(c). $n \equiv 2 \pmod 4$.
Let $n = 4k + 2$ and $k \in N$. As in case 3 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 2)$ and v_i $(1 \le i \le 4k + 2)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(4) = 4k + r + 3$ and $t_{p\varphi}(3) = 4k + r + 4$.

Subcase 11(d). $n \equiv 3 \pmod 4$.
Let $n = 4k + 3$ and $k \in N$. As in case 4 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 3)$ and v_i $(1 \le i \le 4k + 3)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = t_{p\varphi}(3) = 4k + r + 4$ and $t_{p\varphi}(4) = 4k + r + 5$.

Case 12. $p + q \equiv 3 \pmod 4$.
Suppose $t_{pf}(1) = t_{pf}(2) = t_{pf}(3) = r + 1$ and $t_{pf}(4) = r$.

Subcase 12(a). $n \equiv 0 \pmod 4$.
Let $n = 4k$ and $k \in N$. As in case 1 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k)$ and v_i $(1 \le i \le 4k)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 2$ and $t_{p\varphi}(2) = 4k + r + 1$.

Subcase 12(b). $n \equiv 1 \pmod 4$.
Let $n = 4k + 1$ and $k \in N$. As in case 2 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 1)$ and v_i $(1 \le i \le 4k + 1)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = 4k + r + 3$ and $t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 2$.

Subcase 12(c). $n \equiv 2 \pmod 4$.
Let $n = 4k + 2$ and $k \in N$. The proof is similar to Subcase 5(c) in Case 5. Then $t_{p\varphi}(1) = t_{p\varphi}(4) = 4k + r + 3$ and $t_{p\varphi}(2) = t_{p\varphi}(3) = 4k + r + 4$.

Subcase 12(d). $n \equiv 3 \pmod 4$.
Let $n = 4k + 3$ and $k \in N$. As in case 4 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 3)$ and v_i $(1 \le i \le 4k + 3)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = 4k + r + 5$ and $t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 4$.

Case 13. $p + q \equiv 3 \pmod 4$.
Suppose $t_{pf}(1) = t_{pf}(2) = t_{pf}(4) = r + 1$ and $t_{pf}(3) = r$.

Subcase 13(a). $n \equiv 0 \pmod 4$.
Let $n = 4k$ and $k \in N$. As in case 1 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k)$ and v_i $(1 \le i \le 4k)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(4) = 4k + r + 2$ and $t_{p\varphi}(2) = t_{p\varphi}(3) = 4k + r + 1$.

The Ciência & Engenharia - Science & Engineering Journal
ISSN: 0103-944X
Volume 11 Issue 1, 2023
pp: 2189 - 2194

Subcase 13(b). $n \equiv 1 \pmod 4$.

Let $n = 4k+1$ and $k \in \mathbb{N}$. The proof is similar to Subcase 4(b) in Case 4. Then $t_{p\varphi}(1) = t_{p\varphi}(4) = 4k + r + 3$ and $t_{p\varphi}(2) = t_{p\varphi}(3) = 4k + r + 2$.

Subcase 13(c). $n \equiv 2 \pmod 4$.
Let $n = 4k+2$ and $k \in \mathbb{N}$. As in case 3 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k+2)$ and v_i $(1 \le i \le 4k+2)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = 4k + r + 4$ and $t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 3$.

Subcase 13(d). $n \equiv 3 \pmod 4$.
Let $n = 4k + 3$ and $k \in \mathbb{N}$. As in case 3 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k + 3)$ and v_i $(1 \le i \le 4k+3)$. Finally we relabel the vertex u_{4k+3} by 3. Let h be defined by the relabeled technique of the bistar $B_{n,n}$ in (1). Let φ be the function as in (1). It is easy to verify that $t_{p\varphi}(1) = t_{p\varphi}(2) = 4k + r + 4$ and $t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 5$.

Case 14. $p + q \equiv 3 \pmod 4$.
Suppose $t_{pf}(1) = t_{pf}(3) = t_{pf}(4) = r + 1$ and $t_{pf}(2) = r$.

Subcase 14(a). $n \equiv 0 \pmod 4$.
Let $n = 4k$ and $k \in \mathbb{N}$. The proof is similar to Subcase 5(a) in Case 5. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = 4k + r + 1$ and $t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 2$.

Subcase 14(b). $n \equiv 1 \pmod 4$.
Let $n = 4k+1$ and $k \in \mathbb{N}$. As in case 2 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k+1)$ and v_i $(1 \le i \le 4k+1)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = 4k + r + 3$ and $t_{p\varphi}(2) = t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 2$.

Subcase 14(c). $n \equiv 2 \pmod 4$.
Let $n = 4k+2$ and $k \in \mathbb{N}$. As in case 3 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k+2)$ and v_i $(1 \le i \le 4k+2)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(3) = 4k + r + 4$ and $t_{p\varphi}(2) = t_{p\varphi}(4) = 4k + r + 3$.

Subcase 14(d). $n \equiv 3 \pmod 4$.
Let $n = 4k+3$ and $k \in \mathbb{N}$. As in case 4 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k+3)$ and v_i $(1 \le i \le 4k+3)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(4) = 4k + r + 5$ and $t_{p\varphi}(2) = t_{p\varphi}(3) = 4k + r + 4$.

Case 15. $p + q \equiv 3 \pmod 4$.
Suppose $t_{pf}(2) = t_{pf}(3) = t_{pf}(4) = r + 1$ and $t_{pf}(1) = r$.

Subcase 15(a). $n \equiv 0 \pmod 4$.
Let $n = 4k$ and $k \in \mathbb{N}$. As in case 1 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k)$ and v_i $(1 \le i \le 4k)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(2) = 4k + r + 1$ and $t_{p\varphi}(3) = t_{p\varphi}(4) = 4k + r + 2$.

Subcase 15(b). $n \equiv 1 \pmod 4$.
Let $n = 4k+1$ and $k \in \mathbb{N}$. As in case 2 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k+1)$ and v_i $(1 \le i \le 4k+1)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(3) = 4k + r + 2$ and $t_{p\varphi}(2) = t_{p\varphi}(4) = 4k + r + 3$.

Subcase 15(c). $n \equiv 2 \pmod 4$.
Let $n = 4k+2$ and $k \in \mathbb{N}$. As in case 3 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k+2)$ and v_i $(1 \le i \le 4k+2)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(4) = 4k + r + 3$ and $t_{p\varphi}(2) = t_{p\varphi}(3) = 4k + r + 4$.

Subcase 15(d). $n \equiv 3 \pmod 4$.
Let $n = 4k+3$ and $k \in \mathbb{N}$. As in case 4 of Theorem 3.1, assign the label to the vertices u, v, u_i $(1 \le i \le 4k+3)$ and v_i $(1 \le i \le 4k+3)$ together with the 4-total prime cordial f of G is also satisfies the condition of 4-total prime cordial labeling of this case. Then $t_{p\varphi}(1) = t_{p\varphi}(3) = 4k + r + 4$ and $t_{p\varphi}(2) = t_{p\varphi}(4) = 4k + r + 5$.

The Ciência & Engenharia - Science & Engineering Journal
ISSN: 0103-944X
Volume 11 Issue 1, 2023
pp: 2189 - 2194

5. Illustration

4-total prime cordial graph of Jelly fish $J_{5,5}$ is given below:

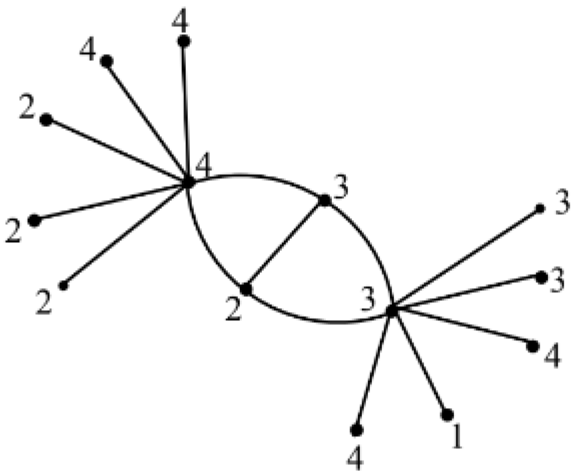

Figure 1

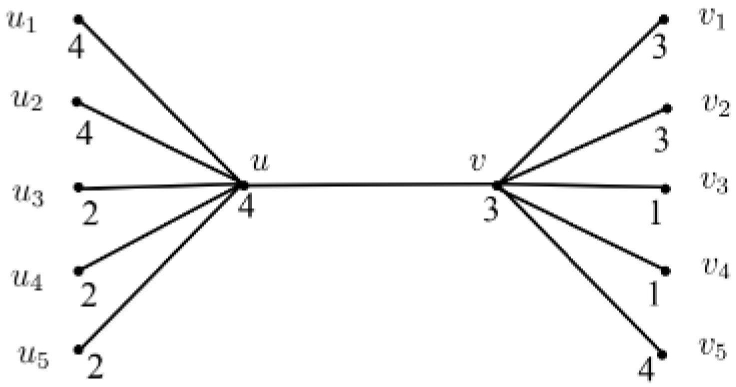

Figure 2. $B_{5,5}$

Change the label of u_4 by 3 and v_5 by 2 as in Subcase 4(b). We get a 4-total prime cordial labeling of $J_{5,5} \cup B_{5,5}$ is shown in Figure 3.

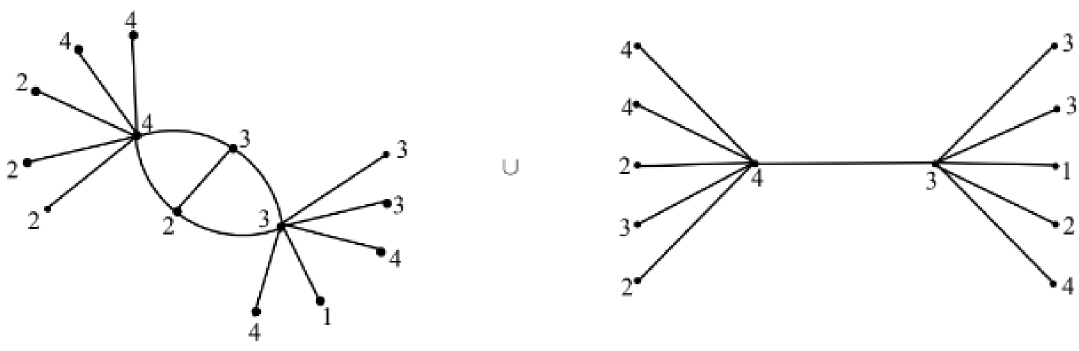

Figure 3. $J_{5,5} \cup B_{5,5}$

6. Conclusion

We have discuss the 4-total prime cordial labeling of $G \cup B_{n,n}$, where G is a 4-total prime cordial graph. The investigation of 4-total prime cordiality of $G \cup H$, G is a 4-total prime cordial graphs and H is any other graphs is an open problem for future research work.

References

[1] Abdullah Aljouiee, On Prime Cordial Labeling of Graphs, *Kyungpook Math. J.*, 56 (2016), 41-46.
[2] I.Cahit, Cordial graphs:A weaker version of graceful and harmonious graphs, *Ars Combinatoria*, 23(1987), 201-207.
[3] J.A.Gallian, A Dynamic survey of graph labeling, *The Electronic Journal of Combinatorics*, 19 (2017) #Ds6.
[4] G.V.Ghodasara and J.P.Jena, Prime Cordial Labeling of the Graphs Related to Cycle with One Chord, Twin Chords and Triangle , *International Journal of Pure and Applied Mathematics*, 89(1), (2013), 79-87.
[5] F.Harary, Graph theory, *Addision wesley*, New Delhi (1969).
[6] R.Ponraj, J.Maruthamani and R.Kala, *k*-Total prime cordial labeling of graphs, *Journal of Algorithms and Computation*, 50(1), 143-149.
[7] R.Ponraj, J.Maruthamani and R.Kala, 4-total prime cordial labeling of some special graphs, *Jordan Journal of Mathematics and Statistics*, 13(1) (2020), 153-168.
[8] R.Ponraj, J.Maruthamani and R.Kala, New Families of 4-total prime cordial graph, *Jordan Journal of Mathematics and Statistics*, 13(4) (2020), 547-563.
[9] R.Ponraj and J.Maruthamani, 4-total prime cordial labeling of some derived graphs, *Ars Combinatoria*, (Accept for Publication).
[10] A.H.Rokad, Product Cordial Labeling of Double Wheel and Double Fan Re- lated Graphs, *Kragujevac Journal of Mathematics*, 43(1) (2019), 7-13.

The Ciência & Engenharia - Science & Engineering Journal
ISSN: 0103-944X
Volume 11 Issue 1, 2023
pp: 2189 - 2205

1. Assistant Professor,
P.G Department of Mathematics, Thiruvalluvar College, Papanasam-627 412,
Tamilnadu, India.
E-mail address: drjmmaths@gmail.com
2. Associate Professor and Head,
P.G and Research Department of Mathematics, Aditanar College of Arts and Science,
Tiruchendur-628 216,
Tamilnadu, India.
E-mail address: pasunkilipandian@yahoo.com