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Additive Inverse Cordial Labeling of Graphs

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Abstract

Let G be a (p,q) graph. Consider the function Ψ from $V(G)$ to the set $0,\pm 1$ and the induced surjective edge function $\Psi_e: E(G) \to \{0,\pm 1,\pm 2\}$ defined by $\Psi_e(uv) = \Psi(u) + \Psi(v)$. Let $v_\Psi(x)$ and $e_\Psi(y)$ respectively denote the number of vertices labelled by x, where $x \in \{0,\pm 1\}$, and number of edges labelled by y, where $y \in \{\pm 1,\pm 2\}$. Then Ψ is said to be an additive inverse cordial labelingif $|v_\Psi(i) - v_\Psi(j)| \leq 1$, where $i,j \in \{0,\pm 1\}$ and $e_\Psi(x) = e_\Psi(-x), x \in \{\pm 1,\pm 2\}$. If G has an additive inverse cordial labeling Ψ then we say that G is an additive inverse cordial graph.

1. Introduction

Let $G = (V,E)$ be a (p,q) graph. Throughout this paper we have considered only simple, connected, and undirected graphs. The number of vertices of G is called the order of G and the number of edges of G is called the size G. Graph labeling has wide range of applications, see [2]. The graph labeling problem was introduced by Rosa called graceful labelling [4] in the year 1967. In 1980, Cahit[1] introduced the cordial labeling of graphs. Motivated by this labelingwe introduce here a new type of graph labeling, called additive inverse cordial labeling. Also here we discuss the nature of this labeling on some standard graphs. Let x be any real number. Then the symbol $\lfloor x \rfloor$ stands for the largest integer less than or equal to x and $\lceil x \rceil$ stands for the smallest integer greater than or equal to x. Terms and definitions not defined here are used in the sense of Harary[3].

2. Preliminaries

Defintion: 2.1

Let u and v be (not necessarily distinct) vertices of a graph G. A u-v walkof G is a finite, alternating sequence $u = u_0, e_1, u_1, e_2, \ldots, u_{n-1}, e_n, u_n = v$ of vertices and edges, beginning with vertex u and ending with vertex v, such that $e_i = u_{i-1}u_i$, for $i = 1,2,\ldots,n$. The number n is called the length of the walk.

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A walk $u_0, e_1, u_1, \ldots, u_{n-1}, e_n, u_n$ is determined by the sequence $u_0, u_1, u_2, \ldots, u_{n-1}, u_n$ of its vertices and hence we specify a walk simply by $(u_0, u_1, \ldots, u_n)$. A walk in which $u_0, u_1, \ldots, u_n$are distinct is called a path. A walk $(u_0, u_1, \ldots, u_n)$ is called a closed walk if $u_0 = u_n$. A closed walk in which $u_0, u_1, \ldots, u_{n-1}$ are distinct is called a cycle. A path on n vertices is denoted by P_n and a cycle on n vertices is denoted by C_n.

Defintion: 2.2$K_{1,n}$is called a star.

Defintion: 2.3A graph G is complete, if every two of its vertices are adjacent. A complete graph on n vertices is denoted by K_n.

Defintion: 2.4 Let G be a (p,q) graph. Consider the function Ψ from V(G) to the set {0, ± 1} and the induced surjective edge function $\Psi_e: E(G) \rightarrow \{0, \pm 1, \pm 2\}$defined by $\Psi_e(uv) = \Psi\ (u) + \Psi(v)$. Let $v_\Psi(x)$ and $e_\Psi(y)$ respectively denote the number of vertices labelled by x, where x∈{0, ± 1}, and number of edges labelled by y, where y∈{± 1, ± 2}. Then Ψ is said to be an additive inverse cordial labeling if $|v_\Psi(i) - v_\Psi(j)| \leq 1$, where i,j∈{0, ± 1} and $e_\Psi(x) = e_\Psi(-x), x \in \{\pm 1, \pm 2\}$. If G has an additive inverse cordial labeling Ψ then we say that G is an additive inverse cordial graph.

Illustration 2.1 The graph given in fig 1. is an additive inverse cordial graph.

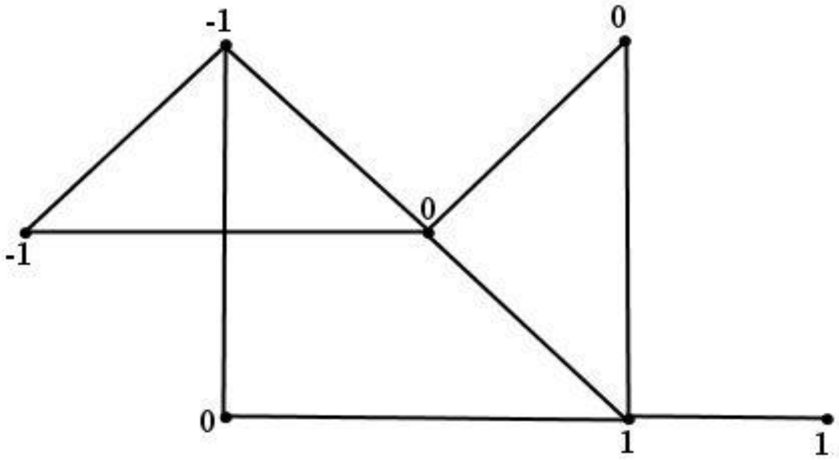

Figure.1

3. Main Results

Theorem: 3.1 The path P_n is additive inverse cordial if and only if n > 4.

Proof. Let $P_n: u_1, u_2 \ldots u_n$be the path. Since Ψ_e is an onto function, we need atleast four edges so that all the four numbers from the set {± 1, ± 2} should be used to label the edges. This implies n > 4. On the other hand, we consider the following cases:

Case 1: $n \equiv\ 0\ (mod\ \ 12)$

Take n=12t, t > 0. Here we define Ψ from V(G) to {0, ± 1} as follows

$$\Psi(u_i) = \begin{cases} -1 & if \quad 1 \le i \le 4t \\ 0 & if \quad 4t+1 \le i \le 8t \\ 1 & if \quad 8t+1 \le i \le 4t \end{cases}$$

Then the induced edge function is given by.

$$\Psi_e(u_i u_{i+1}) = \begin{cases} -2 & if \quad 1 \le i \le 4t-1 \\ -1 & if \quad i = 4t \\ 0 & if \quad 4t+1 \le i \le 8t-1 \\ 1 & if \quad i = 8t \\ 2 & if \quad 8t+1 \le i \le 12t-1 \end{cases}$$

Here $v_\Psi(-1) = v_\Psi(1) = v_\Psi(0) = 4t, e_\Psi(-2) = e_\Psi(2) = 4t-1 \; and \; e_\Psi(-1) = e_\Psi(1) = 1.$

Case 1: $n \equiv 1 \; (mod \;\; 12)$

Take n=12t+1, t > 0. Define a map $\Psi : V(G) \to \{0, \pm 1\}$ by

$$\Psi(u_i) = \begin{cases} -1 & if \quad 1 \le i \le 4t+1 \\ 0 & if \quad 4t+2 \le i \le 8t+1 \\ 1 & if \quad 8t+2 \le i \le 4t+2 \end{cases}$$

Then the induced edge function is given by.

$$\Psi_e(u_i u_{i+1}) = \begin{cases} -2 & if \quad 1 \le i \le 4t \\ -1 & if \quad i = 4t+1 \\ 0 & if \quad 4t+2 \le i \le 8t+1 \\ 1 & if \quad i = 8t+2 \\ 2 & if \quad 8t+3 \le i \le 12t+4 \end{cases}$$

Here $v_\Psi(-1) = v_\Psi(1) = 4t, v_\Psi(0) = 4t+1, e_\Psi(-2) = e_\Psi(2) = 4t-1 \; and \; e_\Psi(-1) = e_\Psi(1) = 1.$

Case 3: $n \equiv 5 \; (mod \;\; 12)$

Take n=12t+2, t > 0. Here we define a function Ψ : V(G) →{0, ± 1} by

$$\Psi(u_i) = \begin{cases} -1 & if \quad 1 \le i \le 4t+1 \\ 0 & if \quad 4t+2 \le i \le 8t+1 \\ 1 & if \quad 8t+2 \le i \le 4t+2 \end{cases}$$

Then the induced edge function becomes

$$\Psi_e(u_i u_{i+1}) = \begin{cases} -2 & if \quad 1 \le i \le 4t \\ -1 & if \quad i = 4t+1 \\ 0 & if \quad 4t+2 \le i \le 8t \\ 1 & if \quad i = 8t+1 \\ 2 & if \quad 8t+2 \le i \le 12t+1 \end{cases}$$

Here v_Ψ(-1)=v_Ψ(1)=4t+1, v_Ψ(0)=4t, e_Ψ(-2)=e_Ψ(2)=4t and e_Ψ (-1)=e_Ψ(1)=1.

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Case 3: $n \equiv 3 \ (mod \ 12)$

Take n=12t+3, t > 0. Here we define a function Ψ : V(G) →{ 0, ± 1} by

$$\Psi(u_i) = \begin{cases} -1 & if \quad 1 \le i \le 4t+1 \\ 0 & if \quad 4t+2 \le i \le 8t+2 \\ 1 & if \quad 8t+3 \le i \le 12t+2 \end{cases}$$

Then the induced edge function Ψ_e becomes

$$\Psi_e(u_i u_{i+1}) = \begin{cases} -2 & if \quad 1 \le i \le 4t \\ -1 & if \quad i = 4t+1 \\ 0 & if \quad 4t+2 \le i \le 8t+1 \\ 1 & if \quad i = 8t+2 \\ 2 & if \quad 8t+3 \le i \le 12t+2 \end{cases}$$

In this case

v_Ψ (-1)=v_Ψ(1)=v_Ψ(0)=4t+1, e_Ψ(-2)=e_Ψ(2)=4t and e_Ψ(-1)=e_Ψ(1)=1.

Case 5: $n \equiv 4 \ (mod \ 12)$

Take n=12t+4, t > 0. Define Ψ : V(G) →{ 0, ± 1} by

$$\Psi(u_i) = \begin{cases} -1 & if \quad 1 \le i \le 4t+1 \\ 0 & if \quad 4t+2 \le i \le 8t+3 \\ 1 & if \quad 8t+4 \le i \le 12t+4 \end{cases}$$

Then the edge function Ψ_e becomes

$$\Psi_e(u_i u_{i+1}) = \begin{cases} -2 & if \quad 1 \le i \le 4t \\ -1 & if \quad i = 4t+1 \\ 0 & if \quad 4t+2 \le i \le 8t+2 \\ 1 & if \quad i = 8t+3 \\ 2 & if \quad 8t+4 \le i \le 12t+3 \end{cases}$$

In this case

v_Ψ(-1)=v_Ψ(1)=4t+1, v_Ψ(0)=4t+2, e_Ψ(-2)=e_Ψ(2)=4t and e_Ψ(-1)=e_Ψ(1)=1.

Case 6: $n \equiv 5 \ (mod \ 12)$

Take n=12t+5, t ≥ 0. Define a map Ψ : V(G) →{0, ± 1} by

$$\Psi(u_i) = \begin{cases} -1 & if \quad 1 \le i \le 4t+2 \\ 0 & if \quad 4t+3 \le i \le 8t+3 \\ 1 & if \quad 8t+4 \le i \le 12t+5 \end{cases}$$

Then the edge function Ψ_e is given by

$$\Psi_e(u_iu_{i+1}) = \begin{cases} -2 \ \ if \ \ 1 \leq i \leq 4t+1 \\ -1 \ \ if \ \ i = 4t+2 \\ 0 \ \ if \ \ 4t+3 \leq i \leq 8t+2 \\ 1 \ \ if \ \ i = 8t+3 \\ 2 \ \ if 8t+4 \leq i \leq 12t+4 \end{cases}$$

Clearly v_Ψ(-1)=v_Ψ(1)=4t+2, v_Ψ(0)=4t+1,e_Ψ(-2)=e_Ψ(2)=4t+1 and

e_Ψ(-1)=e_Ψ(1)=1.

Case 7: $n \equiv \ 6 \ (mod \ \ 12)$

Take n=12t+6, t ≥ 0. Define a map Ψ from V(G) to the set{0, ± 1} as follows

$$\Psi(u_i) = \begin{cases} -1 \ \ if \ \ 1 \leq i \leq 4t+2 \\ 0 \ \ if \ \ 4t+3 \leq i \leq 8t+4 \\ 1 \ if \ \ 8t+5 \leq i \leq 12t+6 \end{cases}$$

Then the function Ψ_e is given by.

$$\Psi_e(u_iu_{i+1}) = \begin{cases} -2 \ \ if \ \ 1 \leq i \leq 4t+1 \\ -1 \ \ if \ \ i = 4t+2 \\ 0 \ \ if \ \ 4t+3 \leq i \leq 8t+3 \\ 1 \ \ if \ \ i = 8t+4 \\ 2 \ \ if \ \ 8t+5 \leq i \leq 12t+5 \end{cases}$$

Here v_Ψ(-1)=v_Ψ(1)=v_Ψ(0)=4t+2, e_Ψ(-2)=e_Ψ(2)=4t+1 and e_Ψ(-1)=e_Ψ(1)=1.

Case 8: $n \equiv \ 7 \ (mod \ \ 12)$

Let n=12t+7, $t \geq$ 0. We define a map Ψ from V(G) to {0, ± 1} by

$$\Psi(u_i) = \begin{cases} -1 \ \ if \ \ 1 \leq i \leq 4t+2 \\ 0 \ \ if \ \ 4t+3 \leq i \leq 8t+5 \\ 1 \ if \ \ 8t+6 \leq i \leq 12t+7 \end{cases}$$

Then the induced edge function $\Psi_e: E(G) \rightarrow \{0, \pm 1, \pm 2\}$ is given by

$$\Psi_e(u_iu_{i+1}) = \begin{cases} -2 \ \ if \ \ 1 \leq i \leq 4t+1 \\ -1 \ \ if \ \ i = 4t+2 \\ 0 \ \ if \ \ 4t+3 \leq i \leq 8t+4 \\ 1 \ \ if \ \ i = 8t+5 \\ 2 \ \ if \ \ 8t+5 \leq i \leq 12t+6 \end{cases}$$

Then v_Ψ(-1)=v_Ψ(1)=4t+2, v_Ψ(0)=4t+3, e_Ψ(-2)=e_Ψ(2)=4t+1 and e_Ψ(-1)=e_Ψ(1)=1.

Case 9: $n \equiv \ 8 \ (mod \ \ 12)$

Let n=12t+8, t ≥ q 0. Define a function Ψ from V(G) to {0, ± 1} by

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$$\Psi(u_i) = \begin{cases} -1 \ \ if \quad 1 \le i \le 4t+3 \\ 0 \ \ if \quad 4t+4 \le i \le 8t+5 \\ 1 \ if \ \ 8t+6 \le i \le 12t+8 \end{cases}$$

Then the edge function Ψ_e is given by

$$\Psi_e(u_i u_{i+1}) = \begin{cases} -2 \ \ if & 1 \le i \le 4t+2 \\ -1 \ \ if & i = 4t+3 \\ 0 \ \ if & 4t+4 \le i \le 8t+4 \\ 1 \ \ if & i = 8t+5 \\ 2 \ \ if & 8t+6 \le i \le 12t+7 \end{cases}$$

Then $v_\Psi(-1) = v_\Psi(1) = 4t+3, v_\Psi(0) = 4t+2, e_\Psi(-2) = e_\Psi(2) = 4t+2$ and $e_\Psi(-1) = e_\Psi(1) = 1$.

Case 10: $n \equiv \ 9 \ (mod \ \ 12)$

Let n=12t+9, t ≥ 0. Define a map Ψ : V(G) →{0, ± 1} as follows

$$\Psi(u_i) = \begin{cases} -1 \ \ if \quad 1 \le i \le 4t+3 \\ 0 \ \ if \quad 4t+4 \le i \le 8t+6 \\ 1 \ if \ \ 8t+7 \le i \le 12t+9 \end{cases}$$

Then the edge function Ψ_e is given by

$$\Psi_e(u_i u_{i+1}) = \begin{cases} -2 \ \ if & 1 \le i \le 4t+2 \\ -1 \ \ if & i = 4t+3 \\ 0 \ \ if & 4t+4 \le i \le 8t+5 \\ 1 \ \ if & i = 8t+6 \\ 2 \ \ if & 8t+7 \le i \le 12t+8 \end{cases}$$

Then $v_\Psi(-1) = v_\Psi(1) = v_\Psi(0) = 4t+3, e_\Psi(-2) = e_\Psi(2) = 4t+2$ and $e_\Psi(-1) = e_\Psi(1) = 1$.

Case 11: $n \equiv \ 10 \ (mod \ \ 12)$

Let n=12t+10, t ≥ 0. Define a function Ψ from V(G) to {0, ± 1} by

$$\Psi(u_i) = \begin{cases} -1 \ \ if \quad 1 \le i \le 4t+3 \\ 0 \ \ if \quad 4t+4 \le i \le 8t+7 \\ 1 \ if \ \ 8t+8 \le i \le 12t+10 \end{cases}$$

Then the induced edge map Ψ_e becomes

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$$\Psi_e(u_iu_{i+1}) = \begin{cases} -2 \ \ if & 1 \le i \le 4t+2 \\ -1 \ \ if & i = 4t+3 \\ \ \ 0 \ \ if & 4t+4 \le i \le 8t+6 \\ \ \ 1 \ \ if & i = 8t+7 \\ \ \ 2 \ \ if & 8t+8 \le i \le 12t+9 \end{cases}$$

In this case $v_\Psi(-1) = v_\Psi(1) = 4t+3, v_\Psi(0) = 4t+4, e_\Psi(-2) = e_\Psi(2) = 4t+2$ and $e_\Psi(-1) = e_\Psi(1) = 1$.

Case 12: $n \equiv 11 \ (mod \ \ 12)$

Let n=12t+11, t ≥ 0. Define Ψ : V(G) →{0, ± 1} by

$$\Psi(u_i) = \begin{cases} -1 \ \ if & 1 \le i \le 4t+4 \\ 0 \ \ if & 4t+5 \le i \le 8t+7 \\ 1 \ if & 8t+8 \le i \le 12t+11 \end{cases}$$

Then the induced edge map Ψ_e becomes

$$\Psi_e(u_iu_{i+1}) = \begin{cases} -2 \ \ if & 1 \le i \le 4t+3 \\ -1 \ \ if & i = 4t+4 \\ \ \ 0 \ \ if & 4t+5 \le i \le 8t+6 \\ \ \ 1 \ \ if & i = 8t+7 \\ 2 \ \ if & 8t+8 \le i \le 12t+10 \end{cases}$$

Here $v_\Psi(-1) = v_\Psi(1) = 4t+3, v_\Psi(0) = 4t+3, e_\Psi(-2) = e_\Psi(2) = 4t+3$ and $e_\Psi(-1) = e_\Psi(1) = 1$.

Corollary:3.1.1 The cycle C_n is additive inverse cordial, if and only if, n >4.

Proof: Since Ψ_e is a surjective function, we need atleast four edges so that all the four labels from the set {± 1, ± 2} should be used to label the edges, so n > 3. Consider the case n=4, suppose there exists an additive inverse cordial labeling Ψ. As Ψ_e is an onto function, -2 should be a label of any of the four edges of the cycle C_n. This label can be obtained only from two adjacent vertices with the label -1. So we have left with two non labeled vertices. As -2 is one of the edge label 2 should be a label of any of the remaining non-labeled edges. It is possible only when the two non-labeled adjacent vertices should be labeled by 1. If so, $v_\Psi(-1) = v_\Psi(1) = 2, v_\Psi(0) = 0$, a contradiction. Hence$C_4$ is not an additive inverse cordial graph. For n > 4, the labeling given in theorem 3.1 satisfies the required conditions. Hence C_n is additive inverse cordial if n > 4.

Theorem: 3.2$K_{1,n}$ is not additive inverse cordial.

Proof. Let u be the vertex of degree n in $K_{1,n}$. Suppose we assign 0 to u, then there doesn't exist an edge with the label 2. If we assign 1 to u, then 2 cannot be an edge label. Suppose we

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choose -1 as label of u, then there does not exist an edge with the label 2. In either case, Ψ_e is not onto and hence $K_{1,n}$ does not admit an additive inverse cordial labeling.

Theorem: 3.3K_n is additive inverse cordial iff n> 3.

Proof. As the edge map is surjective, n should be greater than 3. Conversely, suppose n ≥ 4 then we define Ψ as follows:

Case 1: $n \equiv\ 0\ (mod\ \ 3)$

Let n=3t, t > 0. Assign the label 0 to t vertices and then -1 to another t vertices and the remaining vertices are labeled by 1. In this case, $v_\Psi(-1) = v_\Psi(1) = v_\Psi(0) = t, e_\Psi(-2) = e_\Psi(2) = \binom{t}{2}$and $e_\Psi(-1) = e_\Psi(1) = t^2$.

Case 2: $n \equiv\ 1\ (mod\ \ 3)$

Let n=3t+1, t > 0. Assign the label -1 to first t vertices then assign 1 to next t vertices. Finally we assign the label 0 to t+1 vertices. Here $v_\Psi(-1) = v_\Psi(1) = t, v_\Psi(0) = t+1, e_\Psi(-2) = e_\Psi(2) = \binom{t}{2}\ and\ e_\Psi(-1) = e_\Psi(1) = (t+1)t.$

Case 3: $n \equiv 2\ (mod\ \ 3)$

Let n=3t+2, t > 0. Assign the label -1 to first t+1 vertices and assign the label 1 to next t+1 vertices. Finally we assign the label 0 to t vertices. Note that $v_\Psi(-1) = v_\Psi(1) = t+1, v_\Psi(0) = t, e_\Psi(-2) = e_\Psi(2) = \binom{t+1}{2}$

$$and\ e_\Psi(-1) = e_\Psi(1) = (t+1)t.$$

Hence K_n is additive inverse cordial for all n ≥ 4.

Example 3.1A complete graph K_9 with an additive inverse cordial labeling is given below.

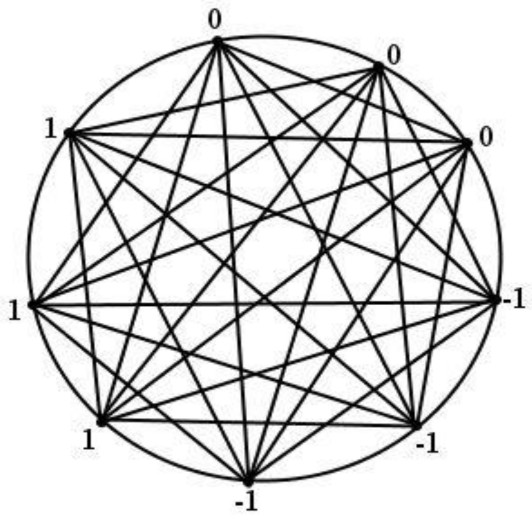

Figure.2

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