Additive Inverse Cordial Labeling of Graphs

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Abstract

Let $G$ be a $(p, q)$ graph. Consider the function $\Psi$ from $V(G)$ to the set $0, \pm 1$ and the induced surjective edge function $\Psi_e: E(G) \rightarrow \{0, \pm 1, \pm 2\}$ defined by $\Psi_e(uv) = \Psi(u) + \Psi(v)$. Let $\nu(x)$ and $\varepsilon(y)$ respectively denote the number of vertices labelled by $x$, where $x \in \{0, \pm 1\}$, and number of edges labelled by $y$, where $y \in \{\pm 1, \pm 2\}$. Then $\Psi$ is said to be an additive inverse cordial labeling if $|\nu(i) - \nu(j)| \leq 1$, where $i, j \in \{0, \pm 1\}$ and $\varepsilon(x) = \varepsilon(-x), x \in \{\pm 1, \pm 2\}$. If $G$ has an additive inverse cordial labeling $\Psi$ then we say that $G$ is an additive inverse cordial graph.

1. Introduction

Let $G = (V, E)$ be a $(p, q)$ graph. Throughout this paper we have considered only simple, connected, and undirected graphs. The number of vertices of $G$ is called the order of $G$ and the number of edges of $G$ is called the size $G$. Graph labeling has wide range of applications, see [2]. The graph labeling problem was introduced by Rosa called graceful labelling [4] in the year 1967. In 1980, Cahit[1] introduced the cordial labeling of graphs. Motivated by this labeling we introduce here a new type of graph labeling, called additive inverse cordial labeling. Also here we discuss the nature of this labeling on some standard graphs. Let $x$ be any real number. Then the symbol $[x]$ stands for the largest integer less than or equal to $x$ and $\lfloor x \rfloor$ stands for the smallest integer greater than or equal to $x$. Terms and definitions not defined here are used in the sense of Harary[3].

2. Preliminaries

Definition: 2.1

Let $u$ and $v$ be (not necessarily distinct) vertices of a graph $G$. A $u$-$v$ walk of $G$ is a finite, alternating sequence $u = u_0, e_1, u_1, e_2, \ldots, u_{n-1}, e_n, u_n = v$ of vertices and edges, beginning with vertex $u$ and ending with vertex $v$, such that $e_i = u_{i-1}u_i$, for $i = 1, 2, \ldots, n$. The number $n$ is called the length of the walk.
A walk $u_0, e_1, u_1, \ldots, u_{n-1}, e_n, u_n$ is determined by the sequence $u_0, u_1, u_2, \ldots, u_{n-1}, u_n$ of its vertices and hence we specify a walk simply by $(u_0, u_1, \ldots, u_n)$. A walk in which $u_0, u_1, \ldots, u_n$ are distinct is called a path. A walk $(u_0, u_1, \ldots, u_n)$ is called a closed walk if $u_0 = u_n$. A closed walk in which $u_0, u_1, \ldots, u_{n-1}$ are distinct is called a cycle. A path on $n$ vertices is denoted by $P_n$ and a cycle on $n$ vertices is denoted by $C_n$.

**Definition:** 2.2 $K_{1,n}$ is called a star.

**Definition:** 2.3 A graph $G$ is complete, if every two of its vertices are adjacent. A complete graph on $n$ vertices is denoted by $K_n$.

**Definition:** 2.4 Let $G$ be a $(p,q)$ graph. Consider the function $\Psi$ from $V(G)$ to the set $\{0, \pm 1\}$ and the induced surjective edge function $\Psi_e : E(G) \rightarrow \{0, \pm 1, \pm 2\}$ defined by $\Psi_e(uv) = \Psi(u) + \Psi(v)$. Let $v_x(x)$ and $e_y(y)$ respectively denote the number of vertices labelled by $x$, where $x \in \{0, \pm 1\}$, and number of edges labelled by $y$, where $y \in \{\pm 1, \pm 2\}$. Then $\Psi$ is said to be an additive inverse cordial labeling if $|v_x(i) - v_x(j)| \leq 1$, where $i, j \in \{0, \pm 1\}$ and $e_y(x) = e_y(-x), x \in \{\pm 1, \pm 2\}$. If $G$ has an additive inverse cordial labeling $\Psi$ then we say that $G$ is an additive inverse cordial graph.

**Illustration 2.1** The graph given in fig 1. is an additive inverse cordial graph.

![Figure 1](https://seer-ufu-br.online)

3. Main Results

**Theorem:** 3.1 The path $P_n$ is additive inverse cordial if and only if $n > 4$.

**Proof.** Let $P_n; u_1, u_2 \ldots u_n$ be the path. Since $\Psi_e$ is an onto function, we need at least four edges so that all the four numbers from the set $\{\pm 1, \pm 2\}$ should be used to label the edges. This implies $n > 4$. On the other hand, we consider the following cases:

**Case 1:** $n \equiv 0 \pmod{12}$

Take $n = 12t$, $t > 0$. Here we define $\Psi$ from $V(G)$ to $\{0, \pm 1\}$ as follows
Then the induced edge function becomes

\[
\Psi(u_i) = \begin{cases} 
-1 & \text{if } 1 \leq i \leq 4t \\
0 & \text{if } 4t + 1 \leq i \leq 8t \\
1 & \text{if } 8t + 1 \leq i \leq 4t 
\end{cases}
\]

Then the induced edge function is given by.

\[
\Psi_e(u_iu_{i+1}) = \begin{cases} 
-2 & \text{if } 1 \leq i \leq 4t - 1 \\
-1 & \text{if } i = 4t \\
0 & \text{if } 4t + 1 \leq i \leq 8t - 1 \\
1 & \text{if } i = 8t \\
2 & \text{if } 8t + 1 \leq i \leq 12t - 1 
\end{cases}
\]

Here \( \nu(1) = \nu(0) = 4t, \nu(-2) = \nu(2) = 4t - 1 \) and \( \nu(-1) = \nu(1) = 1. \)

**Case 1:** \( n \equiv 1 \mod 12 \)

Take \( n=12t+1, t > 0 \). Define a map \( \Psi : V(G) \to \{0, \pm 1\} \) by

\[
\Psi(u_i) = \begin{cases} 
-1 & \text{if } 1 \leq i \leq 4t + 1 \\
0 & \text{if } 4t + 2 \leq i \leq 8t + 1 \\
1 & \text{if } 8t + 2 \leq i \leq 4t + 2 
\end{cases}
\]

Then the induced edge function is given by.

\[
\Psi_e(u_iu_{i+1}) = \begin{cases} 
-2 & \text{if } 1 \leq i \leq 4t \\
-1 & \text{if } i = 4t + 1 \\
0 & \text{if } 4t + 2 \leq i \leq 8t + 1 \\
1 & \text{if } i = 8t + 2 \\
2 & \text{if } 8t + 3 \leq i \leq 12t + 4 
\end{cases}
\]

Here \( \nu(1) = \nu(0) = 4t, \nu(0) = 4t + 1, \nu(-2) = \nu(2) = 4t - 1 \) and \( \nu(-1) = \nu(1) = 1. \)

**Case 3:** \( n \equiv 5 \mod 12 \)

Take \( n=12t+2, t > 0 \). Here we define a function \( \Psi : V(G) \to \{0, \pm 1\} \) by

\[
\Psi(u_i) = \begin{cases} 
-1 & \text{if } 1 \leq i \leq 4t + 1 \\
0 & \text{if } 4t + 2 \leq i \leq 8t + 1 \\
1 & \text{if } 8t + 2 \leq i \leq 4t + 2 
\end{cases}
\]

Then the induced edge function becomes

\[
\Psi_e(u_iu_{i+1}) = \begin{cases} 
-2 & \text{if } 1 \leq i \leq 4t \\
-1 & \text{if } i = 4t + 1 \\
0 & \text{if } 4t + 2 \leq i \leq 8t \\
1 & \text{if } i = 8t + 1 \\
2 & \text{if } 8t + 2 \leq i \leq 12t + 1 
\end{cases}
\]

Here \( \nu(1) = 4t + 1, \nu(0) = 4t, \nu(-2) = \nu(2) = 4t - 1 \) and \( \nu(-1) = \nu(1) = 1. \)
Case 3: $n \equiv 3 \pmod{12}$

Take $n = 12t + 3$, $t > 0$. Here we define a function $\Psi : V(G) \to \{0, \pm 1\}$ by

$$
\Psi(u_i) = \begin{cases} 
-1 & \text{if } 1 \leq i \leq 4t + 1 \\
0 & \text{if } 4t + 2 \leq i \leq 8t + 2 \\
1 & \text{if } 8t + 3 \leq i \leq 12t + 2 
\end{cases}
$$

Then the induced edge function $\Psi_e$ becomes

$$
\Psi_e(u_i u_{i+1}) = \begin{cases} 
-2 & \text{if } 1 \leq i \leq 4t \\
-1 & \text{if } i = 4t + 1 \\
0 & \text{if } 4t + 2 \leq i \leq 8t + 1 \\
1 & \text{if } i = 8t + 2 \\
2 & \text{if } 8t + 3 \leq i \leq 12t + 2 
\end{cases}
$$

In this case

$v_{\Psi}(-1) = v_{\Psi}(1) = v_{\Psi}(0) = 4t + 1$, $e_{\Psi}(-2) = e_{\Psi}(2) = 4t$ and $e_{\Psi}(-1) = e_{\Psi}(1) = 1$.

Case 5: $n \equiv 4 \pmod{12}$

Take $n = 12t + 4$, $t > 0$. Define $\Psi : V(G) \to \{0, \pm 1\}$ by

$$
\Psi(u_i) = \begin{cases} 
-1 & \text{if } 1 \leq i \leq 4t + 1 \\
0 & \text{if } 4t + 2 \leq i \leq 8t + 3 \\
1 & \text{if } 8t + 4 \leq i \leq 12t + 4 
\end{cases}
$$

Then the edge function $\Psi_e$ becomes

$$
\Psi_e(u_i u_{i+1}) = \begin{cases} 
-2 & \text{if } 1 \leq i \leq 4t \\
-1 & \text{if } i = 4t + 1 \\
0 & \text{if } 4t + 2 \leq i \leq 8t + 2 \\
1 & \text{if } i = 8t + 3 \\
2 & \text{if } 8t + 4 \leq i \leq 12t + 3 
\end{cases}
$$

In this case

$v_{\Psi}(-1) = v_{\Psi}(1) = 4t + 1$, $v_{\Psi}(0) = 4t + 2$, $e_{\Psi}(-2) = e_{\Psi}(2) = 4t$ and $e_{\Psi}(-1) = e_{\Psi}(1) = 1$.

Case 6: $n \equiv 5 \pmod{12}$

Take $n = 12t + 5$, $t \geq 0$. Define a map $\Psi : V(G) \to \{0, \pm 1\}$ by

$$
\Psi(u_i) = \begin{cases} 
-1 & \text{if } 1 \leq i \leq 4t + 2 \\
0 & \text{if } 4t + 3 \leq i \leq 8t + 3 \\
1 & \text{if } 8t + 4 \leq i \leq 12t + 5 
\end{cases}
$$

Then the edge function $\Psi_e$ is given by
Clearly $v_\Psi(-1)=v_\Psi(1)=4t+2$, $v_\Psi(0)=4t+1, e_\Psi(-2)=e_\Psi(2)=4t+1$ and $e_\Psi(-1)=e_\Psi(1)=1$.

**Case 7: $n \equiv 6 \pmod{12}$**

Take $n=12t+6, t \geq 0$. Define a map $\Psi$ from $V(G)$ to the set $\{0, \pm 1\}$ as follows

$$\Psi(u_i) = \begin{cases} 
-1 & \text{if } 1 \leq i \leq 4t+2 \\
0 & \text{if } 4t+3 \leq i \leq 8t+4 \\
1 & \text{if } 8t+5 \leq i \leq 12t+6
\end{cases}$$

Then the function $\Psi_e$ is given by

$$\Psi_e(u_iu_{i+1}) = \begin{cases} 
-2 & \text{if } 1 \leq i \leq 4t+1 \\
-1 & \text{if } i = 4t+2 \\
0 & \text{if } 4t+3 \leq i \leq 8t+3 \\
1 & \text{if } i = 8t+4 \\
2 & \text{if } 8t+5 \leq i \leq 12t+5
\end{cases}$$

Here $v_\Psi(-1)=v_\Psi(1)=v_\Psi(0)=4t+2, e_\Psi(-2)=e_\Psi(2)=4t+1$ and $e_\Psi(-1)=e_\Psi(1)=1$.

**Case 8: $n \equiv 7 \pmod{12}$**

Let $n=12t+7, t \geq 0$. Define a map $\Psi$ from $V(G)$ to $\{0, \pm 1\}$ by

$$\Psi(u_i) = \begin{cases} 
-1 & \text{if } 1 \leq i \leq 4t+2 \\
0 & \text{if } 4t+3 \leq i \leq 8t+5 \\
1 & \text{if } 8t+6 \leq i \leq 12t+7
\end{cases}$$

Then the induced edge function $\Psi_e: E(G) \rightarrow \{0, \pm 1, \pm 2\}$ is given by

$$\Psi_e(u_iu_{i+1}) = \begin{cases} 
-2 & \text{if } 1 \leq i \leq 4t+1 \\
-1 & \text{if } i = 4t+2 \\
0 & \text{if } 4t+3 \leq i \leq 8t+4 \\
1 & \text{if } i = 8t+5 \\
2 & \text{if } 8t+5 \leq i \leq 12t+6
\end{cases}$$

Then $v_\Psi(-1)=v_\Psi(1)=4t+2, v_\Psi(0)=4t+3, e_\Psi(-2)=e_\Psi(2)=4t+1$ and $e_\Psi(-1)=e_\Psi(1)=1$.

**Case 9: $n \equiv 8 \pmod{12}$**

Let $n=12t+8, t \geq 0$. Define a function $\Psi$ from $V(G)$ to $\{0, \pm 1\}$ by
Then the edge function $\Psi_e$ is given by

$$\Psi_e(u_i u_{i+1}) = \begin{cases} 
-2 & \text{if } 1 \leq i \leq 4t + 2 \\
-1 & \text{if } i = 4t + 3 \\
0 & \text{if } 4t + 4 \leq i \leq 8t + 4 \\
1 & \text{if } i = 8t + 5 \\
2 & \text{if } 8t + 6 \leq i \leq 12t + 7 
\end{cases}$$

Then $v_\Psi (-1) = v_\Psi (1) = 4t + 3, v_\Psi (0) = 4t + 2, e_\Psi (-2) = e_\Psi (2) = 4t + 2$ and $e_\Psi (-1) = e_\Psi (1) = 1$.

**Case 10: $n \equiv 9 \pmod{12}$**

Let $n=12t+9$, $t \geq 0$. Define a map $\Psi : V(G) \rightarrow \{0, \pm 1\}$ as follows

$$\Psi(u_i) = \begin{cases} 
-1 & \text{if } 1 \leq i \leq 4t + 3 \\
0 & \text{if } 4t + 4 \leq i \leq 8t + 6 \\
1 & \text{if } 8t + 7 \leq i \leq 12t + 9 
\end{cases}$$

Then the edge function $\Psi_e$ is given by

$$\Psi_e(u_i u_{i+1}) = \begin{cases} 
-2 & \text{if } 1 \leq i \leq 4t + 2 \\
-1 & \text{if } i = 4t + 3 \\
0 & \text{if } 4t + 4 \leq i \leq 8t + 5 \\
1 & \text{if } i = 8t + 6 \\
2 & \text{if } 8t + 7 \leq i \leq 12t + 8 
\end{cases}$$

Then $v_\Psi (-1) = v_\Psi (1) = v_\Psi (0) = 4t + 3, e_\Psi (-2) = e_\Psi (2) = 4t + 2$ and $e_\Psi (-1) = e_\Psi (1) = 1$.

**Case 11: $n \equiv 10 \pmod{12}$**

Let $n=12t+10$, $t \geq 0$. Define a function $\Psi$ from $V(G)$ to $\{0, \pm 1\}$ by

$$\Psi(u_i) = \begin{cases} 
-1 & \text{if } 1 \leq i \leq 4t + 3 \\
0 & \text{if } 4t + 4 \leq i \leq 8t + 7 \\
1 & \text{if } 8t + 8 \leq i \leq 12t + 10 
\end{cases}$$

Then the induced edge map $\Psi_e$ becomes
In this case \( v_\Psi(-1) = v_\Psi(1) = 4t + 3, v_\Psi(0) = 4t + 4, e_\Psi(-2) = e_\Psi(2) = 4t + 2 \) and \( e_\Psi(-1) = e_\Psi(1) = 1 \).

**Case 12:** \( n \equiv 11 \pmod{12} \)

Let \( n = 12t + 11, t \geq 0 \). Define \( \Psi : V(G) \rightarrow \{0, \pm 1\} \) by

\[
\Psi(u_i) = \begin{cases} 
-1 & \text{if } 1 \leq i \leq 4t + 4 \\
0 & \text{if } 4t + 5 \leq i \leq 8t + 7 \\
1 & \text{if } 8t + 8 \leq i \leq 12t + 11 
\end{cases}
\]

Then the induced edge map \( \Psi_e \) becomes

\[
\Psi_e(u_i u_{i+1}) = \begin{cases} 
-2 & \text{if } 1 \leq i \leq 4t + 3 \\
-1 & \text{if } i = 4t + 4 \\
0 & \text{if } 4t + 5 \leq i \leq 8t + 6 \\
1 & \text{if } i = 8t + 7 \\
2 & \text{if } 8t + 8 \leq i \leq 12t + 10 
\end{cases}
\]

Here \( v_\Psi(-1) = v_\Psi(1) = 4t + 3, v_\Psi(0) = 4t + 3, e_\Psi(-2) = e_\Psi(2) = 4t + 3 \) and \( e_\Psi(-1) = e_\Psi(1) = 1 \).

**Corollary 3.1.1** The cycle \( C_n \) is additive inverse cordial, if and only if, \( n > 4 \).

**Proof:** Since \( \Psi_e \) is a surjective function, we need at least four edges so that all the four labels from the set \( \{\pm 1, \pm 2\} \) should be used to label the edges, so \( n > 3 \). Consider the case \( n = 4 \), suppose there exists an additive inverse cordial labeling \( \Psi \). As \( \Psi_e \) is an onto function, \( -2 \) should be a label of any of the four edges of the cycle \( C_n \). This label can be obtained only from two adjacent vertices with the label -1. So we have left with two non-labeled vertices. As -2 is one of the edge label 2 should be a label of any of the remaining non-labeled edges. It is possible only when the two non-labeled adjacent vertices should be labeled by 1. If so, \( v_\Psi(-1) = v_\Psi(1) = 2, v_\Psi(0) = 0 \), a contradiction. Hence \( C_4 \) is not an additive inverse cordial graph. For \( n > 4 \), the labeling given in theorem 3.1 satisfies the required conditions. Hence \( C_n \) is additive inverse cordial if \( n > 4 \).

**Theorem 3.2** \( K_{1,n} \) is not additive inverse cordial.

**Proof.** Let \( u \) be the vertex of degree \( n \) in \( K_{1,n} \). Suppose we assign 0 to \( u \), then there doesn't exist an edge with the label 2. If we assign 1 to \( u \), then 2 cannot be an edge label. Suppose we
choose -1 as label of u, then there does not exist an edge with the label 2. In either case, $\Psi_e$ is not onto and hence $K_{1,n}$ does not admit an additive inverse cordial labeling.

**Theorem: 3.3** $K_n$ is additive inverse cordial iff $n \geq 3$.

**Proof.** As the edge map is surjective, $n$ should be greater than 3. Conversely, suppose $n \geq 4$ then we define $\Psi$ as follows:

**Case 1: $n \equiv 0 \pmod{3}$**
Let $n=3t$, $t > 0$. Assign the label 0 to $t$ vertices and then -1 to another $t$ vertices and the remaining vertices are labeled by 1. In this case, $v_\Psi(-1) = v_\Psi(1) = v_\Psi(0) = t, e_\Psi(-2) = e_\Psi(2) = \binom{t}{2}$ and $e_\Psi(-1) = e_\Psi(1) = t^2$.

**Case 2: $n \equiv 1 \pmod{3}$**
Let $n=3t+1$, $t > 0$. Assign the label -1 to first $t$ vertices then assign 1 to next $t$ vertices. Finally we assign the label 0 to $t+1$ vertices. Here $v_\Psi(-1) = v_\Psi(1) = t, v_\Psi(0) = t + 1, e_\Psi(-2) = e_\Psi(2) = \binom{t}{2}$ and $e_\Psi(-1) = e_\Psi(1) = (t + 1)t$.

**Case 3: $n \equiv 2 \pmod{3}$**
Let $n=3t+2$, $t > 0$. Assign the label -1 to first $t+1$ vertices and assign the label 1 to next $t+1$ vertices. Finally we assign the label 0 to $t$ vertices. Note that $v_\Psi(-1) = v_\Psi(1) = t + 1, v_\Psi(0) = t, e_\Psi(-2) = e_\Psi(2) = \binom{t+1}{2}$

and $e_\Psi(-1) = e_\Psi(1) = (t + 1)t$.

Hence $K_n$ is additive inverse cordial for all $n \geq 4$.

**Example 3.1** A complete $K_9$ with an additive inverse cordial labeling is given below.

![Figure 2](https://seer-ufu-br.online)
References