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New Vistas on Mersenne numbers

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Abstract:
Mersenne numbers are analyzed for varieties of interesting properties. Various fascinating relations connecting Mersenne numbers with other special number patterns by means of theorems involving the relations are exhibited.

Introduction:
Number is the essence of mathematical calculations. Numbers have varieties of patterns and have varieties of range and richness. Many numbers exhibit fascinating properties, they form sequences, they form patterns and so on. In this context one may refer [1-11]. This communication presents varieties of fascinating properties on Mersenne numbers. Also, many beautiful relations connecting Mersenne numbers with other special number patterns by means of theorems involving the relations are presented.

Notations:

- Polygonal number of rank n with sides m:

$$t_{m,n} = n\left[1 + \frac{(n-1)(m-2)}{2}\right]$$

- Jacobsthal number:

$$J_n = \frac{2^n - (-1)^n}{3}$$

- Jacobsthal-Lucas number:

$$j_n = 2^n + (-1)^n$$

- Carol number:

$$carl_n = \left(2^n - 1\right)^2 - 2$$

- Kynea number:

$$Ky_n = \left(2^n + 1\right)^2 - 2$$

- Icosahedral number:

$$I_n = \frac{n\left(5n^2 - 5n + 2\right)}{2}$$

- Thebit ibn Kurrah number:

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$$TH_n = 3.2^n - 1$$

Properties:

Theorem1: $M_{n+k} = M_n M_k + M_n + M_k$

Proof:

$$M_{n+k} = 2^{n+k} - 1 = (2^n - 1)(2^k - 1) + 2^n + 2^k - 2$$
$$= (2^n - 1)(2^k - 1) + (2^n - 1) + (2^k - 1)$$
$$= M_n M_k + M_n + M_k$$

Theorem2: $M_n^2 = M_{2n} + 2(M_n - M_{n+1} + 1)$

Proof:

$$M_n^2 = (2^n - 1)^2 = 2^{2n} - 2^{n+1} + 1 = (2^{2n} - 1) + 2(1 - 2^n)$$
$$= (2^{2n} - 1) + 2[(2^n - 1) - (2^{n+1} - 1) + 1]$$
$$= M_{2n} + 2(M_n - M_{n+1} + 1)$$

Theorem3: $\mathrm{M}_{n+1}^2 - 4\mathrm{M}_n^2 = \mathrm{M}_{n+2} - 1$

Proof:

$$M_{n+1}^2 - 4M_n^2 = (2^{n+1} - 1)^2 - 4(2^n - 1)^2 = 42^n - 3$$
$$= 2^{n+2} - 2 = \mathrm{M}_{n+2} - 1$$

Theorem4: $M_{3n} = M_n(M_{2n} + M_n + 3)$

Proof: $M_{3n} = 2^{3n} - 1 = (2^n - 1)(2^{2n} + 2^n + 1)$

$$= (2^n - 1)[(2^{2n} - 1) + 2^n + 2]$$
$$= (2^n - 1)[(2^{2n} - 1) + (2^n - 1) + 3]$$
$$= M_n(M_{2n} + M_n + 3)$$

Theorem5: $M_{n+k}^2 = M_{2n+2k} - M_{n+k+1} + 1$

$$= 2^{2n+2k} - 2^{n+k+1} + 1$$
$$= (2^{2n+2k} - 1) - (2^{n+k+1} - 1) + 1$$
$$= M_{2n+2k} - M_{n+k+1} + 1$$

Theorem6: $2^k M_{n-k} M_{n+k} = (M_n - M_k)(M_n + M_k + M_n M_k)$

Proof:

$$2^k M_{n-k} M_{n+k} = 2^k(2^{n-k} - 1)(2^{n+k} - 1)$$
$$= (2^n - 2^k)(2^{n+k} - 1)$$
$$= [(2^n - 1) - (2^k - 1)][(2^n - 1)(2^k - 1) + 2^n + 2^k - 2]$$
$$= (M_n - M_k)[M_n M_k + M_n + M_k]$$

Theorem7: $M_{2n} M_{2n+k} = M_k(M_{4n} - M_{2n}) + M_{4n} - 2M_{2n}$

Proof: $M_{2n} M_{2n+k} = (2^{2n} - 1)(2^{2n+k} - 1)$

$$= 2^k(2^{4n} - 2^{2n}) - (2^{2n} - 1)$$
$$= [(2^k - 1) + 1][(2^{4n} - 1) - (2^{2n} - 1)] - M_{2n}$$

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$$=(M_k+1)(M_{4n}-M_{2n})-M_{2n}$$
$$=M_k(M_{4n}-M_{2n})+(M_{4n})-2M_{2n}$$

Theorem8: $\sum_{i=1}^{n} M_{i+k}^2 = \frac{1}{9}(TH_{2k+2}+1)M_{2n} - \frac{1}{3}(TH_{k+2}+1)M_n + n$

Proof:

$$\sum_{i=1}^{n} M_{i+k}^2 = \sum_{i=1}^{n} \left(2^{i+k}-1\right)^2$$
$$= 2^{2k}\sum_{i=1}^{n} 2^{2i} - 2^{k+1}\sum_{i=1}^{n} 2^{i} + n$$
$$= \frac{2^{2k+2}}{3} M_{2n} - 2^{k+2} M_n + n$$
$$= \frac{1}{9}(TH_{2k+2}+1)M_{2n} - \frac{1}{3}(TH_{k+2}+1)M_n + n$$

Theorem 9: $\sum_{i=1}^{n} (M_{k+i}+1)M_{3i} = \frac{(TH_k+1)}{45}(16M_{4n}-30M_n)$

Proof:

$$M_{3i}M_{k+i} = \left(2^{3i}-1\right)\left(2^{k+i}-1\right)$$
$$= 2^k\left(2^{4i}-2^i\right) - \left(2^{3i}-1\right)$$
$$= 2^k\left(2^{4i}-2^i\right) - M_{3i}$$
$$\therefore \quad M_{3i}(M_{k+i}+1) = 2^k\left(2^{4i}-2^i\right)$$

$$\sum_{i=1}^{n} (M_{k+i}+1)M_{3i} = 2^k \sum_{i=1}^{n}\left(2^{4i}-2^i\right)$$
$$= 2^k\left(\sum_{i=1}^{n} 2^{4i} - \sum_{i=1}^{n} 2^i\right)$$
$$= 2^k\left(\frac{16\left(2^{4n}-1\right)}{15} - 2\left(2^n-1\right)\right)$$
$$= \frac{2^k}{15}(16M_{4n}-30M_n)$$
$$= \frac{3.2^k}{45}(16M_{4n}-30M_n)$$
$$= \frac{(TH_k+1)}{45}(16M_{4n}-30M_n)$$

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Theorem 10: $\sum_{i=1}^{n}\left(M_{i+k}-M_{i}\right)=2M_{k}M_{n}$

Proof:

$$M_{i+k}-M_{i}=2^{i+k}-2^{i}$$
$$=\left(2^{k}-1\right)2^{i}$$
$$=M_{k}2^{i}$$
$$\therefore \sum_{i=1}^{n}\left(M_{i+k}-M_{i}\right)=M_{k}\sum_{i=1}^{n}2^{i}=2M_{k}M_{n}$$

Theorem 11: $\sum_{i=1}^{n}M_{i}M_{3i}=\frac{8}{15}j_{2n}\left(4t_{3,M_{n}}-\frac{1}{7}M_{n}\right)-\frac{16}{7}t_{3,M_{n}}-2M_{n}+n$

Proof:

$$\sum_{i=1}^{n}M_{i}M_{3i}=\sum_{i=1}^{n}\left(2^{i}-1\right)\left(2^{3i}-1\right)$$
$$=\sum_{i=1}^{n}2^{4i}-\sum_{i=1}^{n}2^{3i}-\sum_{i=1}^{n}2^{i}+\sum_{i=1}^{n}1$$
$$=\frac{16}{15}\left(2^{4n}-1\right)-\frac{8}{7}\left(2^{3n}-1\right)-2\left(2^{n}-1\right)+n$$
$$=\frac{16}{15}\left(2^{2n}+1\right)\left(2^{n}-1\right)\left(2^{n}+1\right)-\frac{8}{7}\left(2^{n}-1\right)\left(2^{2n}+2^{n}+1\right)-2M_{n}+n$$
$$=\frac{16}{15}j_{2n}\left(2^{n}\left(2^{n}-1\right)+\left(2^{n}-1\right)\right)-\frac{8}{7}\left(M_{n}j_{2n}+\left(2^{n}-1\right)2^{n}\right)-2M_{n}+n$$
$$=\frac{16}{15}j_{2n}\left(2t_{3,M_{n}}+M_{n}\right)-\frac{8}{7}\left(M_{n}j_{2n}+2t_{3,M_{n}}\right)-2M_{n}+n$$
$$=\frac{32}{15}j_{2n}t_{3,M_{n}}+j_{2n}M_{n}\left(\frac{16}{15}-\frac{8}{7}\right)-\frac{16}{7}t_{3,M_{n}}-2M_{n}+n$$
$$=\frac{8}{15}j_{2n}\left(4t_{3,M_{n}}-\frac{1}{7}M_{n}\right)-\frac{16}{7}t_{3,M_{n}}-2M_{n}+n$$

Theorem 12: $M_{2n}^{2}=M_{n}^{4}+4M_{n}^{2}\left(3J_{n}+(-1)^{n}\right)$

Proof:

$$M_{2n}^{n}=\left(2^{2n}-1\right)^{2}=\left(2^{n}-1\right)^{2}\left(2^{n}+1\right)^{2}$$
$$=\left(2^{n}-1\right)^{2}\left(\left(2^{n}-1\right)^{2}+4.2^{n}\right)$$
$$=\left(2^{n}-1\right)^{4}+4.2^{n}\left(2^{n}-1\right)^{2}$$
$$=\left(2^{n}-1\right)^{4}+4\left(2^{n}-1\right)^{2}\left(2^{n}-(-1)^{n}+(-1)^{n}\right)$$
$$=M_{n}^{4}+4M_{n}^{2}\left(3J_{n}+(-1)^{n}\right)$$

Theorem 13: $M_{2n}^2 = M_n^2\left(Ky_n + 2\right)$

Proof:

$$\begin{aligned} M_{2n}^2 &= \left(2^{2n}-1\right)^2 = \left(2^n-1\right)^2\left(2^n+1\right)^2 \\ &= \left(2^n-1\right)^2\left[\left(2^n+1\right)^2 - 2 + 2\right] \\ &= M_n^2\left(Ky_n + 2\right) \end{aligned}$$

Theorem 14: $M_{2n}^2 = Ky_n * carl_n + 4j_{2n} - 4$

Proof:

$$\begin{aligned} M_{2n}^2 &= \left(2^n-1\right)^2\left(2^n+1\right)^2 \\ &= \left[\left(2^n-1\right)^2 - 2 + 2\right]\left[\left(2^n+1\right)^2 - 2 + 2\right] \\ &= \left[\left(2^n-1\right)^2 - 2\right]\left[\left(2^n+1\right)^2 - 2\right] + 2\left[\left(2^n+1\right)^2 + \left(2^n-1\right)^2 - 2 - 2\right] + 4 \\ &= \left[\left(2^n-1\right)^2 - 2\right]\left[\left(2^n+1\right)^2 - 2\right] + 4\left(2^{2n}\right) \\ &= Ky_n + carl_n + 4j_{2n} - 4 \end{aligned}$$

Theorem 15: $M_{2n}\left(M_{2n+3} + 9\right) + 8 = 8\left(j_{2n} - 1\right)^2$

Proof:

$$\begin{aligned} M_{2n}M_{2n+3} &= \left(2^{2n}-1\right)\left(2^{2n+3}-1\right) \\ &= 8.2^{4n} - 9.2^{2n} + 1 \\ &= 8\left(2^{2n}\right)^2 - 9\left(2^{2n}-1\right) - 8 \\ &= 8\left(j_{2n}-1\right)^2 - 9M_{2n} - 8 \end{aligned}$$

$$M_{2n}\left(M_{2n+3} + 9\right) + 8 = 8\left(j_{2n} - 1\right)^2$$

Theorem 16: $M_{2n+3} + 9 = 8j_{2n}$

Proof:

$$\begin{aligned} M_{2n+3} + 9 &= \left(2^{2n+3}-1\right) + 9 \\ &= 8.2^{2n} + 8 \\ &= 8j_{2n} \end{aligned}$$

Theorem 17: $Ky_{2n} + 4carl_{2n} + M_{2n} + 8 = 2^{1-2n} I_{2^{2n}}$

Proof:

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$$Ky_{2n} + 4carl_{2n} + M_{2n} + 8 = \left(2^{2n}+1\right)^2 - 2 + 4\left[\left(2^{2n}-1\right)^2 - 2\right] + 2^{2n} - 1 + 8$$
$$= 5.2^{4n} - 5.2^{2n} + 2$$
$$= 5.\left(2^{2n}\right)^2 - 5\left(2^{2n}\right) + 2$$
$$= \frac{2}{2^{2n}}\left[\frac{2^{2n}}{2}\left(5.\left(2^{2n}\right)^2 - 5.2^{2n} + 2\right)\right]$$
$$= 2^{1-2n} I_{2^{2n}}$$

Theorem 18: The triple $\left(M_{2n}, 2(M_n+1), M_{2n}+2\right)$ forms a Pythagorean triple. The area A_n and perimeter P_n of the above Pythagorean triangle are given by

$A_n = 6P^3_{M_n}$, $P_n = 4t_{3,M_n+1}$

Proof:

$$M_{2n}^2 + 4(M_n+1)^2 = \left(2^{2n}-1\right)^2 + 4.2^{2n} = \left(2^{2n}+1\right)^2 = (M_{2n}+2)^2$$

Therefore the triple $\left(M_{2n}, 2(M_n+1), M_{2n}+2\right)$ is a Pythagorean triple.

Area:

$$A_n = M_{2n}(M_n+1) = 2^n\left(2^{2n}-1\right)$$
$$= \left(2^n-1\right)2^n\left(2^n+1\right) = 6P^3_{M_n}$$

Perimeter:

$$P_n = 2(M_{2n}+M_n+2) = 2\left(2^{2n}+2^n\right)$$
$$= 2*2^n*\left(2^n+1\right) = 2(M_n+1)(M_n+2)$$
$$= 4t_{3,M_n+1}$$

Note that

I. $P_n + 1 = 2\left(2^{2n}+2^n\right)+1 = 2^{2n} + \left(2^n+1\right)^2$
$= (M_n+1)^2 + (M_n+2)^2$

II. $\dfrac{2A_n}{P_n} = M_n$

III. $\dfrac{2A_n}{P_n} + M_{n+1} + 1 = TH_n$

IV. $\sum_{i=1}^{n} P_i = 2\left(\sum_{i=1}^{n} 2^{2i} + \sum_{i=1}^{n} 2^i\right) = \dfrac{1}{3}\left(24J_{2n} + 4TH_n - 8\right)$

V. $\sum_{i=1}^{n} A_i = \sum_{i=1}^{n} 2^{3i} - \sum_{i=1}^{n} 2^i = \dfrac{1}{7}\left[6 + \dfrac{2}{3}(TH_n+1)(TH_{2n}+3J_{2n}-5)\right]$

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Theorem 19: The triple $\left(M_{2n}+1-(M_k+1)^2\ ,\ 2(M_n+1)(M_k+1),\ M_{2n}+1+(M_k+1)^2\right)$ forms a Pythagorean triple. The area $A_{n,k}$ and the perimeter $P_{n,k}$ of the above Pythagorean triangle are given by

$$P_{n,k}=4t_{3,M_n+1}+2M_k(M_n+1)$$

$$A_{n,k}=(M_{k+1}+1)P^5_{M_n+1}-(M_{n+1}+1)P^5_{M_k+1}+M_{n+2k}-M_{2n+k}$$

Proof:

$$\left[M_{2n}+1-(M_k+1)^2\right]^2+\left[2(M_n+1)(M_k+1)\right]^2$$

$$=\left(2^{2n}-2^{2k}\right)^2+\left(2.2^n.2^k\right)^2=\left(2^{2n}+2^{2k}\right)^2$$

$$=\left[M_{2n}+1+(M_k+1)^2\right]^2$$

Therefore the triple $\left(M_{2n}+1-(M_k+1)^2\ ,\ 2(M_n+1)(M_k+1),\ M_{2n}+1+(M_k+1)^2\right)$ forms a Pythagorean triple.

Perimeter:

$$P_{n,k}=\left(2^n+2^k\right)^2+2^{2n}-2^{2k}=2.\left(2^{2n}+2^{n+k}\right)$$

$$=2.2^n\left(2^n+1\right)+2.2^n.\left(2^k-1\right)$$

$$=4t_{3,M_n+1}+2M_k(M_n+1)$$

Area:

$$A_{n,k}=2^{n+k}\left(2^{2n}-2\right)$$

$$=2^k.2^{3n}-2^n.2^{3k}$$

$$=2^{k+1}P^5_{M_n+1}-(M_{2n+k}+1)-2^{n+1}P^5_{M_k+1}+(M_{n+2k}+1)$$

$$=2^{k+1}P^5_{M_n+1}-2^{n+1}P^5_{M_k+1}-M_{2n+k}+M_{n+2k}$$

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